A Frictional Matching Model of Two-Sided Platforms

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Abstract

Based on the frictional matching framework, the paper provides a theoretical model for the two-sided platform in which the number of participants on both sides of the platform and the source of network externalities are endogenously determined. The platform is shown to exhibit both positive and negative network externalities: the utility of a participant is increasing (decreasing) in the number of participants on the other (same) side of the platform. Unlike the past literature, a participant’s utility is bounded, even if the number of participants on the other side of the platform goes to infinity. The optimal prices of the platform owner are shown to depend not only on costs, but also on the effect that a new entrant (either a buyer or a seller) has on the matching probability.
1 Introduction

Recently there has been a surge in the interests regarding theoretical issues of transaction platform.\(^1\) Basically, a platform is viewed as centralized ground in which economic agents (generally on two side of the platform) can search, trade, or interact. A central feature of the transaction platform is that it exhibits positive network effects: The greater the number of other agents that one can potentially interact with, the greater benefit he can gain from participating in the platform. Almost all of the theoretical models in the literature capture this externalities by assuming that the benefit an agent gains from participating in one side of the platform is a linear function of the number of the agents on the other side of the platform. This setup, though convenient, has several drawbacks.

First, the setup implies that an agent’s utility will grow without bound as the number of agents on the other side goes to infinity. However, unlike the case of the network products from which the consumers derive utilities from the number of users (e.g. see Arthur, 1991), the network externalities in the platforms are indirect. For example, in the online auction platforms, a bidder of an item has higher expected utility when there are more sellers of this item not because the number of sellers per SE, but only because he can obtain the item with higher probability or with lower price. His valuation of the remains same regardless of the number of sellers. Therefore, the utility of a buyer or seller, or in general a participant in a platform, cannot derive infinite utility (as is implicitly assumed in the literature) even when the number of potential agents who can interact with him goes to infinity. Second, the setup also implies that, given the number of agents on the other side of the platform, an agent’s utility remains fixed as the number of the agents on the same side grows. However,

in general a platform not only exhibits positive network externalities, but also negative network externalities as well. Again take the online auction platform as an example. Given the number of sellers online, the probability that a bidder can win an item (reps. the price he will pay when he buys) will be decreasing (reps. increasing) in the number of buyers. In other words, although the number of participants of the other side of the market has positive externalities on the benefit of a participants, the number of participants on the same side will have as adverse effect on his benefit. This is a fact that has not been taken into consideration in the previous literature. There two features in the previous literature will in particular imply that is no optimal size of the transaction platform: the most efficient transaction platform will be one in which both sides have infinite number of agents.

In order to theoretically investigate the two issues mentioned above, it calls for a model which can provide a microfoundation of how the platform functions, so that the value of network externalities (both positive and negative) can be endogenized. In this paper we provide such a theoretical model. The model is adapted from a frictioned matching framework for price-determination by Burdett et al. (2001). In the model, a group of sellers (each having one unit of a good) meet a group buyers (each needing one unit of the good) in a platform. The owner of the platform charges each agent for using the platform. The sellers post prices, and the buyers choose the probability of visiting each seller. A seller can sell his good (at the price he posts) if and only if at least one buyer visits his store. A buyer, if he is the only visitor of a seller, buys the good with probability one. Otherwise the probability that he can buy the good from the seller he visits is 1 over the number of buyers visiting that seller. Prices set by the platform owner determine how many buyers and sellers there will be in the platform. We then solve for the equilibrium prices of both the platform owner and the sellers, together with the equilibrium number of the sellers and buyers and their utilities. A buyer’s utility is shown to be increasing (decreasing) in
the number of sellers (buyers). Similarly, a seller’s utility is increasing (decreasing) in the number of the buyers (sellers). The optimal price of the platform owner is shown to be depend not only on costs, but also on the effect that a new entrant (either a buyer or a seller) has on the matching probability. We therefore endogenize the value of network externalities in a platform with a price-matching framework.

2 Transactions and Frictional Matching

2.1 The Model

Consider the market of a good in which risk neutral buyers trade with risk neutral sellers on a monopoly platform.

Each seller has one unit, and each buyer needs one unit, of the good. The prices charged by the platform owner determine how many buyers, denoted by $N_b$, and how many sellers, denoted by $N_s$, enter the platform. Assume that the price is in the form of membership fees so that when a seller or a buyer joins the platform, he pays a fee of $F_s$ (seller) or $F_b$ (buyer). Each seller posts a price for the good, which every buyer observes. The buyers then determine the probability that he will visit each shop. Ties (more than one buyer intends to buy the good from the same seller) are resolved by randomization of equal chance. Under the setup, role that a platform plays is, on the one hand, to provide price information to the buyers and, on the other hand, to match the buyers and sellers. Note that even if a seller post the lowest price, if there are other sellers who post the same price, he might not be able to sell his good. Similarly, even if a buyer accepts the price of a seller, if there are other buyers who want to buy from the same seller, he might not be able to

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2 Since in our model the buyers and the sellers make transaction at most once, membership fee and transaction fee (commission charged by the platform every time an agent makes transaction) are perfect substitution (when transaction fee is discounted by a factor of transaction probability).
buy the good. In other words, this is a price matching model with friction.

The buyers are homogeneous regarding the valuation of the good, but are heterogeneous in transaction costs. Let \( v^b \) be each buyer’s valuation of the good. The heterogeneous of costs is captured by a parameter of “location” are, \( x^b \), which is uniformly distributed over a line with length \( N \), so \( x^b \sim U(0,N) \) with mass 1. Sellers’ reservation price are \( v^s \) and they are also heterogeneous in their locations, \( x^s \), where \( x^s \sim U(0,N) \) with mass 1. For simplicity, we take the location of the platform as given on 0. In order to travel to the platform, the buyer and the seller pay a unit transportation cost, \( t^b \) and \( t^s \), respectively.\(^3\)

A buyer’s utility function is therefore

\[
U^b = \begin{cases} 
  v^b - p - F^b - t^b x^b, & \text{if he buys the good at price } p; \\
  -F^b - t^b x^b, & \text{if he fails to buy the good; } \\
  0, & \text{if he does not join the platform.}
\end{cases}
\]

Similarly, a seller’s utility function is

\[
U^s = \begin{cases} 
  p - v^s - F^s - t^s x^s, & \text{if he sells the good at price } p; \\
  -F^s - t^s x^s, & \text{if he fails to sell the good; } \\
  0, & \text{if he does not join the platform.}
\end{cases}
\]

The platform’s profit maximizing problem is

\[
\max_{F^b, F^s} \pi = F^b N^b + F^s N^s - c,
\]

where \( c \) is the fixed cost to setup a platform.\(^4\)

Timing of events can be summarized as follow. Stage 1: the platform decides its membership fees (\( F^s \) and \( F^b \)). Stage 2: each seller and each buyer decides whether or not

\(^3\) We assume that \( v^b - v^s \geq \frac{1}{\ln 2} t^b \) and \( v^b - v^s \geq 16 t^s \). This assumption ensures at least two agents of each side will connect with the platform.

\(^4\) As long as the platform has been setup. The marginal cost of serving an extra agent, which has been assume to be zero for simplicity, is relatively small.
to enter the platform (by incurring membership fees). Stage 3: each seller on the platform posts a price. Stage 4: each buyer on the platform chooses a probability of visiting each seller on that platform. We define stage 1 and 2 as the pricing stages and stage 3 and 4 as the frictional matching stages. In the following sections, we will solve for the equilibrium backwardly.

2.2 Frictional Matching Stage

In the frictional matching model in Burdett et al. (2001), there is a unique symmetric equilibrium such that every buyer visit each seller with the same probability, and all sellers post the same price. In our model, there is also a symmetric equilibrium:

**Proposition 1.** Given $N^b$ buyers and $N^s$ sellers in the platform, the symmetric equilibrium has every buyer visit each seller with probability $\frac{1}{N^s}$. Every seller posts the same price

\[ p^* = \frac{v^b[1 - (1 + \frac{N^b}{N^s - 1})(1 - \frac{1}{N^s})^N^b]}{1 - [1 + \frac{N^b}{N^s(N^s - 1)}](1 - \frac{1}{N^s})^N^b} \]

\[ + \frac{v^s \frac{N^b}{N^s}(1 - \frac{1}{N^s})^N^s}{1 - [1 + \frac{N^b}{N^s(N^s - 1)}](1 - \frac{1}{N^s})^N^b}. \]  

Note that (1) in Proposition 1 can be rewritten as $p^* = v^s + (v^b - v^s)z$, where

\[ z = \frac{1 - (1 + \frac{N^b}{N^s - 1})(1 - \frac{1}{N^s})^N^b}{1 - [1 + \frac{N^b}{N^s(N^s - 1)}](1 - \frac{1}{N^s})^N^b}. \]

The net benefit of a match between a buyer and a seller is $v \equiv v^b - v^s$.\(^5\) Therefore, the value of $z$ decides the allocation of this benefit between the seller and the buyer. Since $z$ is increasing in $N^b$ and decreasing in $N^s$,\(^6\) this implies that the seller’s optimal price is increasing in $N^b$ and decreasing in $N^s$. In other words, the seller’s gain from a transaction is increasing (decreasing) in the number of buyers (sellers), a fact that is consistent with

\(^5\) In the matching stage, $F^s$ and $F^b$ are already sunk costs.

\(^6\) See the proof of proposition 2 in the Appendix.
our claim that a platform exhibits both positive and negative network externalities. In the equilibrium, the expected number of matches of this game is

\[ M = N^s [1 - (1 - \frac{1}{N^s})^{N^b}], \tag{3} \]

which is increasing in both \( N^b \) and \( N^s \). The probability of a successful match (or arrival rate) for a buyer and a seller are, respectively, \( A^b = M/N^b \) and \( A^s = M/N^s \). Combine these arrival rates and the equilibrium price we have the expected utility functions of a seller and a buyer in the platform,

\[ \bar{U}^b = v(1 - z)A^b - F^b - t^b x^b, \tag{4} \]
\[ \bar{U}^s = vzA^s - F^s - t^s x^s. \tag{5} \]

It is clear that the expected utilities are functions of the allocation of the benefit from trade, the arrival rate and the cost of joining the platform. The key point is that the values of these two functions depend on the number of buyers and sellers. In particular, a seller’s expected utility is decreasing in \( N^s \) and increasing in \( N^b \), a buyer’s expected utility is increasing in \( N^s \) and decreasing in \( N^b \), as are summarized in Proposition 2.

**Proposition 2.** In equilibrium, expected utility functions exhibit positive cross-group externalities and negative within-group externalities. That is, \( \frac{\partial \bar{U}^b}{\partial N^s} > 0, \frac{\partial \bar{U}^s}{\partial N^s} > 0, \frac{\partial \bar{U}^b}{\partial N^b} < 0, \frac{\partial \bar{U}^s}{\partial N^b} < 0. \)

Proposition 2 indicates that there are both positive and negative network externalities regarding the number of agents in the platform. When the number of agents increases on one side of the platform, the agents on the other side of the platform will have more chance of a successful matching, and therefore a higher expected utility. However, the agents on the same side will face more competition. Consequently, their expected utilities decrease. Most importantly, unlike the previous theoretical literature, these externalities are endogenously
derived, rather assumed, in the model. Also note that when an additional buyer enters, gain of a seller comes not only from the higher arrival rate but also the higher equilibrium price. Furthermore, the buyer’s and the seller’s utilities, $\bar{U}^b$ and $\bar{U}^s$, are bounded even if the number of participants on the other side of the platform goes to infinity:

**Proposition 3.** The willingness to pay for entering the platform of a buyer, $u^b \equiv v(1 - z)A^b$, and a seller, $u^s \equiv vzA^s$, are bounded by the value $v$ as the participants of the other side goes to infinity.

## 3 The Stage of Pricing

Given the equilibrium outcomes in the frictional matching stage derived in the previous section, in this section we will derive the optimal pricing strategy of the platform owner, and the equilibrium number of buyers and sellers ($N^b$ and $N^s$) implied by the owner’s optimal strategy.

The expected utilities computed in (4), (5) are not only the expected utilities of the agents, but also serve as their reservation utilities. That is, at the pricing stage, the seller’s (buyer’s) expected utility has to be at least 0 for him to join the platform willingly. Note that these exists a buyer who locates on $\hat{x}^b$ such that $\bar{U}^b(\hat{x}^b) = 0$. Naturally, buyers with their expected utility greater than or equal to 0 (that is, buyers with $x^b \leq \hat{x}^b$) will join the platform. Given the membership fees of the platform, the number of buyers of the platform is

$$N^b = \hat{x}^b = \frac{1}{\theta^b}(u^b - F^b).$$

The same reasoning applies to the seller’s side, so that

$$N^s = \hat{x}^s = \frac{1}{\theta^s}(u^s - F^s).$$
In the profit maximizing problem of the platform, the first-order conditions of the profit maximizing problem imply that

\[ F^b = t^b N^b - u^b_b N^b - u^s_b N^s, \]  
\[ F^s = t^s N^s - u^s_s N^s - u^b_s N^b, \]

where \( u^b_k = \frac{\partial u^b}{\partial k} \) and \( u^s_k = \frac{\partial u^s}{\partial k} \), \( k \in \{b, s\} \).

If we follow Armstrong (2006), the equilibrium membership fees are \( F^b = t^b N^b - \alpha^s N^s \) and \( F^s = t^s N^s - \alpha^b N^b \), where \( \alpha^b > 0, \alpha^s > 0 \) are the parameters of cross-group externalities to buyers and sellers. Armstrong states that the effects of cross-group externalities, \(-\alpha^s N^s\) and \(-\alpha^b N^b\), adjusted the equilibrium fees downward. We capture these effects by \(-u^s_b N^s\) and \(-u^b_s N^b\).\(^7\) Moreover, in our model, we also capture the effects of within-group externalities by \(-u^b_b N^b\) and \(-u^s_s N^s\). These effects reversely adjust the equilibrium fees upward for we had already proving that \( u^b_b \leq 0 \) and \( u^s_s \leq 0 \) in Proposition 2. One can check that the model in Armstrong (2006) with the within-group externalities been exogenously assumed will have a similar result, either. However, that model can not explain where the externalities come from and how they work if both positive and negative externalities are taken into consideration at the same time. Fortunately, the endogenesis of the externalities in our model help us to figure out these problems.

**Proposition 4.** The equilibrium number of participants of buyers and sellers satisfy

\[ N^{bs} = \frac{v M_b}{2 t^b}, \quad \text{and} \]
\[ N^{ss} = \frac{v M_s}{2 t^s}, \]

\(^7\) Recall that \( u^b_b = \frac{\partial C^b}{\partial N^b} \) and \( u^s_s = \frac{\partial C^s}{\partial N^s} \) are the measures of cross-group externalities in our model.
The profit maximizing membership fees satisfy

\[ F^{bs} = \frac{1}{2} u^b + \frac{1}{2} (u^b - vM_b), \quad \text{and} \]
\[ F^{ss} = \frac{1}{2} u^s + \frac{1}{2} (u^s - vM_s). \]

where

\[ M_b = \frac{\partial M}{\partial N^b} = -N^s(1 - \frac{1}{N^s})^{N^b} \ln(1 - \frac{1}{N^s}) > 0, \quad \text{and} \]
\[ M_s = \frac{\partial M}{\partial N^s} = 1 - (1 + \frac{N^b}{N^s - 1})(1 - \frac{1}{N^s})^{N^b} > 0. \]

From Proposition 4, we can see clearly what are the factors that influence the platform owner’s optimal pricing policy. First, and obviously the participant’s willingness to pay, \( u^k \), will influence the optimal prices. More importantly, price changes also affect the number of either the buyers or the sellers that are willing to enter the platform. This in turn changes the matching probability. The platform thus have to balance the tradeoff between the direct change in income as a consequence of price change and the indirect effect on income brought by the change in the number of the buyers and the sellers. In particular, if \( M_k \) is low, then matching probability is less sensitive to change in the number of participants. In that case the platform is able to charge a higher fees, other things being equal. Similarly, the platform can charge higher fee to the side which has smaller impact on matching probability when the number of its members changes.

If we construct a model with two markets where agents’ willingness to pay, \( u^k \), are independent of each other, then the optimal fees to each market are a half of an agent’s willingness to pay. However, the optimal fees in our model is a half of an agent’s willingness to pay plus \( \frac{1}{2}(u^k - vM_k) \). In other words, the term, \( \frac{1}{2}(u^k - vM_k) \), comes from the interdependence of agents’ willingness to pay. We can verify that \( \frac{1}{2}(u^s - vM_s) \) is positive, however, \( \frac{1}{2}(u^b - vM_b) \) can be either positive or negative.\(^8\) Therefore, we conclude that the

\(^8\) For example, \( \frac{1}{2}(u^b - vM_b) \approx -0.01 \) if \( N^{bs} = 3 \) and \( N^{ss} = 2 \). \( \frac{1}{2}(u^b - vM_b) \approx 0.03 \) if \( N^{bs} = 2 \) and...
monopoly platform might subsidize the buyer-side in the sense that $F^{b*} < \frac{1}{2}u^b$.

4 Conclusion

In this paper we provide a theoretical model of two-sided platforms in which the number of buyers and sellers and, more importantly, the sources of network externalities are endogenously determined. The platform is shown to exhibit both positive and negative network externalities: A participant’s benefit in joining the platform is increasing in the number of participants on the other side of the platform, and is decreasing in the number of participant on the same side. Moreover, unlike the setup in the past literature, the benefit of a participant is bounded, even if the number of participants on the other side of platform goes to infinity. The optimal pricing policy of the platform owner is shown to depend not only on costs but, more importantly, also on the effect that a new entrant (either a buyer or a seller) has on the probability of realized matches. This paper considers only the monopoly platforms. For future research, it will be interesting to investigate the oligopoly case with our framework. In particular, since our model provides a microfoundation for the platform, issues that one difficult to be tackled in the previous theoretical literature such as single-vs. multi-homing choice can be more satisfactorily analyzed.

$N^{**} = 2$. 
References


Appendix

The Proof of Proposition 1

We are going to show that the equilibrium in the frictional matching subgame has every buyer visit each seller with probability $\frac{1}{N^b}$ and every seller post $p^*$.

Follow Burdett et al.(2001), let $\phi$ be the probability that at least one buyer visits a particular seller when all buyer visit him with probability $a$. Then, $\phi = 1 - (1 - a)^{N^b}$, given there are $N^b$ buyers in the platform. Let $\Omega$ be the probability that a given buyer gets served when he visit this seller. Hence

$$\Omega = \frac{\phi}{N^b a} = \frac{1 - (1 - a)^{N^b}}{N^b a}.$$  

If every seller is positing $p$ and one contemplates deviating to $p^d$. The buyer visits the deviant with probability $a^d$. The probability that he visit each of the nondeviants is $\frac{1-a^d}{N^s-1}$, given there are $N^s$ sellers in the platform. So

$$\Omega^d = \frac{1 - (1 - a^d)^{N^b}}{N^b a^d},$$

and a buyer who visits a nondeviant gets served with probability

$$\Omega = \frac{1 - (1 - \frac{1-a^d}{N^s-1})^{N^b}}{N^b(\frac{1-a^d}{N^s-1})}.$$  

In the equilibrium,

$$(v^b - p)\Omega = (v^b - p^d)\Omega^d.$$  

This condition can be written as

$$\frac{v^b - p}{v^b - p^d} = \frac{(1 - a^d)[1 - (1 - a^d)^{N^b}]}{(N^s - 1)a^d[1 - (1 - \frac{1-a^d}{N^s-1})^{N^b}]}.$$  

(14)

Because expected profit of the deviant is $(p^d - v^s)[1 - (1 - a^d)^{N^b}]$, the first-order condition of deviant’s utility maximize problem is

$$[1 - (1 - a^d)^{N^b}] + (p^d - v^s)N^b(1 - a^d)^{N^b-1}\frac{\partial a^d}{\partial p^d} = 0.$$
If we focus on the interior solution such that \( a^d \in (0, 1) \), we can differentiate (14) and then insert the symmetric equilibrium conditions \( p^d = p, a^d = \frac{1}{N^a} \) to derive
\[
\frac{\partial a^d}{\partial p^d} = -\frac{(N^s - 1)[1 - (1 - \frac{1}{N^b})N^b]}{(N^s)^2[(N^s - 1) - (N^s - 1 + N^b)(1 - \frac{1}{N^b})N^b]}(v^b - p^d) < 0.
\]
Inserting this into the first-order condition and solving, we arrive at
\[
p^* = \frac{v^b[1 - (1 + \frac{N^b}{N^b-1})(1 - \frac{1}{N^a})N^a]}{1 - [1 + \frac{N^b}{N^b(N^a-1)}(1 - \frac{1}{N^a})N^a] + \frac{v^s N^b}{N^b} (1 - \frac{1}{N^a})N^a}.
\]

The Proof of Proposition 2

We are going to show that cross-group externalities are positive and within-group externalities are negative. That is,
\[
\frac{\partial \bar{U}^b}{\partial N^s} = v[-z^s A^b + (1 - z)A^b_s] > 0, \\
\frac{\partial \bar{U}^s}{\partial N^b} = v(z^b A^s + z A^b_s) > 0, \\
\frac{\partial \bar{U}^b}{\partial N^b} = v[-z^b A^b + (1 - z)A^b_b] < 0, \\
\frac{\partial \bar{U}^s}{\partial N^s} = v(z^s A^s + z A^s_s) < 0,
\]
where \( z_k \equiv \frac{\partial z}{\partial k}, A^b_k \equiv \frac{\partial A^b}{\partial k} \) and \( A^s_k \equiv \frac{\partial A^s}{\partial k} \), for \( k \in \{b, s\} \).

First, we deal with the more complex part of this proof, the sign of \( z_k \). Let \( z^s = 1 - (1 + \frac{N^b}{N^s-1})(1 - \frac{1}{N^s})N^s \) and \( z^b = \frac{N^b}{N^a}(1 - \frac{1}{N^s})N^b \), then
\[
z = \frac{z^s}{z^s + z^b} = [1 + (\frac{z^s}{z^b})^{-1}]^{-1},
\]
where \( \frac{z^s}{z^b} = \frac{N^s}{N^s}(1 - \frac{1}{N^s})^{-N^b} - 1 - \frac{N^b}{N^s-1} \). The partial derivatives of \( z \) can be written as
\[
\frac{\partial z}{\partial (\frac{z^s}{z^b})} \frac{\partial (\frac{z^s}{z^b})}{\partial N^b}, \\
\frac{\partial z}{\partial (\frac{z^s}{z^b})} \frac{\partial (\frac{z^s}{z^b})}{\partial N^s},
\]

14
where $\frac{\partial z}{\partial (z^s)} = (z^s + 1)^{-2} > 0$. Therefore, we know that $z_b > 0$ if and only if $\frac{\partial (z^s)}{\partial N^b} > 0$, and $z_s < 0$ if and only if $\frac{\partial (z^s)}{\partial N^s} < 0$. However,

$$\frac{\partial (z^s)}{\partial N^b} = -\frac{N^s}{(N^b)^2} (1 - \frac{1}{N^s})^{-N^b} \left[-1 + (1 - \frac{1}{N^s})^{N^b} - \ln(1 - \frac{1}{N^s})^{N^b}\right] > 0,$$

$$\frac{\partial (z^s)}{\partial N^s} = \left[N^b(N^s - 1)(1 - \frac{1}{N^s})^{N^b}\right]^{-1} \times \{N^s[1 - (1 - \frac{1}{N^s})^{N^b}] - N^b - [1 - (1 - \frac{1}{N^s})^{N^b} - \frac{N^b}{N^s - 1}(1 - \frac{1}{N^s})^{N^b}]\} < 0.$$

To verify these, one should know that, first, our assumption such that $N^s \geq 2$ and $N^b \geq 2$ ensures that $\ln(1 - \frac{1}{N^s})$ is well defined and $(1 - \frac{1}{N^s})^{N^b} \in [0, 1]$, second, the term $N^s[1 - (1 - \frac{1}{N^s})^{N^b}] = M \leq \min\{N^b, N^s\}$ by definition. As a result we conclude that $z_b > 0$, $z_s < 0$.

Second, the easier part, we find the sign of $A^b_k$ and $A^s_k$ and they are

$$A^b_k = -\frac{N^s}{(N^b)^2} [1 - (1 - \frac{1}{N^s})^{N^b} + (1 - \frac{1}{N^s})^{N^b} \ln(1 - \frac{1}{N^s})^{N^b}] < 0,$$

$$A^s_k = \frac{1}{N^b} \left[1 - (1 - \frac{1}{N^s})^{N^b} - \frac{N^b}{N^s - 1}(1 - \frac{1}{N^s})^{N^b}\right] > 0,$$

$$A^b_k = -(1 - \frac{1}{N^s})^{N^b} \ln(1 - \frac{1}{N^s}) > 0,$$

$$A^s_k = -\frac{N^b}{N^s(N^s - 1)} (1 - \frac{1}{N^s})^{N^b} < 0.$$

Using these results, one can check the sign of all external effects easily.

The Proof of Proposition 3

To proof this proposition, We are going to show that $A^b$ converges to 1 and $z$ converges to 0 as $N^s$ goes to infinity, $A^s$ converges to 1 and $z$ converges to 1 as $N^b$ goes to infinity. According to the definition of $z$, it is very easy to see that $z$ converges to 0 as $N^s$ goes to infinity and converges to 1 as $N^b$ goes to infinity, so we omit these. To show the limits of $A^b$ and $A^s$, we show that $M$ converges to $N^s$ as $N^b$ goes to infinity and converges to $N^b$ as $N^s$ goes to infinity. Because it is quite obvious that $M$ converges to $N^s$ as $N^b$ goes to
infinity, we omit it. So the only thing that is needed to be proofed is $M$ converges to $N^b$ as $N^s$ goes to infinity. By L’Hôpital’s rule,

$$\lim_{N^s \to \infty} N^s[1 - (1 - \frac{1}{N^s})] = \frac{\lim_{N^s \to \infty} -N^b(1 - \frac{1}{N^s})^{N^b-1} \frac{1}{(N^s)^2}}{\lim_{N^s \to \infty} -\frac{1}{(N^s)^2}} = N^b.$$

The Proof of Proposition 4

Total differentiate $N^b$ and $N^s$, we have

$$\begin{bmatrix} t^b - u^b_s & -u^b_s \\ -u^s_b & t^b - u^s_s \end{bmatrix} \begin{bmatrix} dN^b \\ dN^s \end{bmatrix} = \begin{bmatrix} -dF^b \\ -dF^s \end{bmatrix}.$$

From this system, we can derive the following

$$\begin{align*}
\frac{\partial N^b}{\partial F^b} &= \frac{-(t^s - u^s_s)}{\Delta}, \\
\frac{\partial N^s}{\partial F^b} &= \frac{-u^s_b}{\Delta}, \\
\frac{\partial N^b}{\partial F^s} &= \frac{-u^b_s}{\Delta}, \\
\frac{\partial N^s}{\partial F^s} &= \frac{-(t^b - u^b_b)}{\Delta},
\end{align*}$$

where $\Delta = (t^b - u^b_b)(t^s - u^s_s) - (u^s_b)(u^b_s)$. The two first-order conditions of platform’s profit maximizing problem are

$$\begin{align*}
\frac{\partial \pi}{\partial F^b} &= N^b + F^b \frac{\partial N^b}{\partial F^b} + F^s \frac{\partial N^s}{\partial F^b} = 0, \quad \text{and} \\
\frac{\partial \pi}{\partial F^s} &= F^b \frac{\partial N^b}{\partial F^s} + N^s + F^s \frac{\partial N^s}{\partial F^s} = 0.
\end{align*}$$

These two first-order conditions can be written as

$$\begin{bmatrix} \frac{\partial N^b}{\partial F^b} & \frac{\partial N^s}{\partial F^b} \\ \frac{\partial N^b}{\partial F^s} & \frac{\partial N^s}{\partial F^s} \end{bmatrix} \begin{bmatrix} F^b \\ F^s \end{bmatrix} = \begin{bmatrix} -N^b \\ -N^s \end{bmatrix}.$$
Solving this system we arrive at (6) and (7). Substitute the partial derivates of $N^k$ w.r.t. $F^k$ into (6) and (7), we have

\[ F^{bs} = t^bN^{bs} - vM_b + u^b, \quad \text{and} \]
\[ F^{ss} = t^sN^{ss} - vM_s + u^s. \]

By definition, $t^bN^b = u^b - F^b$ and $t^sN^s = u^s - F^s$, therefore,

\[ F^{bs} = \frac{1}{2}u^b + \frac{1}{2}(u^b - vM_b), \quad \text{and} \]
\[ F^{ss} = \frac{1}{2}u^s + \frac{1}{2}(u^s - vM_s), \]

and

\[ N^{bs} = \frac{vM_b}{2t^b}, \quad \text{and} \]
\[ N^{ss} = \frac{vM_s}{2t^s}. \]