Dynamic Buy-Price English Auction

Kong-Pin Chen*
Research Center of Humanities
and Social Sciences
Academia Sinica
and
Department of Economics
National Taiwan University
kongpin@gate.sinica.edu.tw

Chien-Fu Chou
Department of Economics
National Taiwan University
cfchou@ntu.edu.tw

Abstract

This paper extends the existing theoretical literature on buy-price auctions to the case when the bidders can observe dropouts. In that case when some bidders leave the auction, the remaining bidders have to update their information and change the threshold auction price at which they are willing to buy out accordingly. We explicitly model the information-updating process during auction and the implied recursive nature of the optimal buy-out strategy. Using a recursive equation linking the value functions for \( n \)- and \( (n + 1) \)-bidder cases, we completely characterize the symmetric dynamic optimal buy-out strategy. It is shown that the bidders will postpone buy-out (by waiting until the auction price is higher) when some of them drop out.

* Corresponding author. Research Center for Humanities and Social Sciences, Academia Sinica, Taipei 11529, Taiwan.
1 Introduction

Recently, there has been substantial progress in our understanding of the buy-price online auctions. The earlier literature has emphasized how bidders’ degrees of risk-aversion affects their strategies and the outcome of bidding.\(^1\) Recent research has been focusing on characterizing the equilibrium strategy (of both bidders and sellers) and its outcome. Under general assumptions on utility function and distribution of bidders’ valuations, Hidvegi et al. (2006) characterize the symmetric equilibrium of the Yahoo!-type buy-price auctions.\(^2\) By imposing constant absolute risk-aversion of the bidders, Reynolds and Wooders (2005) characterize the symmetric equilibrium strategies of the bidders in both Yahoo!-type and eBay-type buy-price auctions. They also show that if seller sets the reservation price and the buy price at the same value, then Yahoo!-type auction yields higher expected profit for the seller. Chen et al. (2006) characterize the equilibrium strategy of the bidders and optimal buy-price of the seller, under the assumptions of constant relative risk-aversion (for both bidders and seller) and uniform distribution of valuation. They show that unless both the seller and the bidders are risk-neutral, buy-price increases seller’s expected utility. They also provide empirical evidence in support of certain theoretical predictions derived under the model.

A common feature of all the literature above is that there is no bidders dropout or, equivalently, bidders cannot observe it, when the standing price rises above the levels some are willing to pay and thus they leave the auction. An important consequence of

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\(^1\) See, e.g., Budish and Takeyama, 2001; Mathews and Katzman, (2006).

\(^2\) A Yahoo!-type buy-price auction is one that allows the bidders to obtain the item by paying the buy-price throughout the auction. In contrast, in an eBay-type auction, bidders have this option only before the auction starts. Once bidders start to bid, this option is revoked.
this assumption is that the bidder’s buy-out strategy will remain the same throughout the auction. This is an assumption that greatly simplifies the derivation of the equilibrium strategy, as it will then be unnecessary to consider the information-updating process (and thus the reaction of the remaining bidders) when some of the bidders drop out. However, in a standard English auction this is an important issue because, unlike the sealed-bid auction, its nature is dynamic. A bidder will remain active only if the prevailing price is lower than his valuation, and will drop out as soon as the price rises above it. When that happens, new information is revealed and the behavior of the remaining bidders should change accordingly. Because the literature assumes that the bidders are unaware when some of them drop out, there is no need for this informational update and the change of strategy implied by it. The equilibrium derived is therefore a Bayesian Nash equilibrium, but not subgame perfect.

Specifically, since buy-out is essentially adopted by the seller to intensify the competition between the bidders, the latter are less prone to buy out when the number of bidders decreases.\(^3\) This implies that, other things being equal, when a certain bidder drops out, the remaining bidders will optimally raise the standing price at which they are willing to buy out. More importantly, when timing his optimal buy-out strategy, a bidder must calculate the case when some bidders drop out as one of its contingencies. As a result, the bidder’s optimal buy-out strategy must be inter-linked with that of all the subgames with fewer bidders.\(^4\)

To give a concrete example of our argument above, consider an auction with four

\(^3\) See Chen et al. (2006).

\(^4\) Krishna (2002) also considers the informational updating process in the standard English auctions when bidders observe dropouts. See his Section 6.3.
bidders. When deriving the optimal buy-out strategy, a bidder must take into consideration the possibility that one bidder drops out before the item is sold (which occurs with positive probability). Under that case his expected payoff will be the equilibrium payoff in the 3-bidder auction. However, as is argued above, since a bidder will have different buy-out strategies in the 3-bidder case and 4-bidder cases, optimal buy-out strategy for the latter will depend on that of the former. Therefore, there should be an equation linking the optimal buy-out strategy in the 4-bidder case with that of the 3-bidder case. By the same logic, the optimal strategy of the 3-bidder case should also depend on that of the 2-bidder case. Consequently, the optimal buy-out strategy should be recursively derived. This dynamic structure has not been captured in the previous literature, and in this sense the previous models have been static ones. They essentially assumes that regardless of their valuations, every bidder believes that all his opponents will stay active until the item is sold.

We need to emphasize that a model which assumes no dropout might not be unreasonable for the on-line auctions, as it is indeed impossible to observe drop-outs. As a result, our model should be viewed as an extension of the existing model to the dynamic context, when the number of bidders is observable, rather than as a more general model for online auction with buy-price. However, even in an online auction, to assume that the number of bidders is constant throughout the auction is only a convenient approximation. A complete model for the case when number of bidders is unobservable should specify a (joint) distribution function for all possible valuations of the bidders. For each possible configuration of valuations a bidder derives the optimal buy-out strategy. For example, if a bidder believes that bidder $i$ valuation is $v^i$, then he should act as if bidder $i$ has dropped
out when the price rises above $v^i$, even if he cannot observe it. His strategy should then be the optimal response to the expected value of all the possible configurations considered. Understandably, this results in very complicated computation that is almost impossible to solve.

In this paper we build up a dynamic auction model with buy-out by explicitly taking into consideration the information updating process during auction, and the recursive nature of buy-out strategy implied by this consideration. We derive a unique symmetric perfect Bayesian equilibrium for the bidders, together with a recursive equation to facilitate its computation. Well after the paper was completed, we were made aware of a similar paper by Klumpp and Ranger (2006). They also consider buy-price auctions with observable dropouts and characterize the symmetric equilibrium. The differences between our model and theirs are as follows. First, their model is more general in their assumption on the bidder’s utility function and the distribution of bidder’s valuations. In the case when more than one bidder proposes buy-out, they assume that each wins with equal probability; while we assume the bidder with highest valuation wins. Our model is more general in the assumption on the seller’s utility, which they assume is risk-neutral while we assume it can be either risk-averse or risk-neutral. Because we impose more stringent assumptions on bidders’ utilities and valuations, we can solve for a closed-form solution of the optimal strategy. More importantly, our derivation is very intuitive, which enables us to see clearly the trade-off involved when a bidder considers when to buy out, and how the optimal strategy balances this trade-off.
2 The Model

The model we consider is essentially the same as in Chen et al. (2006). In order to concentrate on the issue raised in the Introduction, we will impose several simplifying assumptions which strip away the complications that are irrelevant to our argument. First, we assume that any bidder’s i’s valuation of the item, $v^i$, is drawn uniformly and independently from the interval $[0, \bar{v}]$. Second, the utility function of the bidder is $(v - p)^{\alpha}/\alpha$, where $v$ is his valuation of the item, $p$ is the price he pays for the item if he wins, and $\alpha \in (0, 1]$ is the (reverse) measure of the bidder’s degree of risk-aversion. The bidder’s utility is 0 if he does not buy the item. Third, when two or more bidders propose to buy out simultaneously, we assume that the one who has higher valuation will win (in buy-out). This assumption not only greatly simplifies our calculation but also affords a closed-form solution of the optimal buy-out strategy. Fourth, we assume that there is no reservation price or, equivalently, the reservation price is 0.

2.1 An Example of the 3-bidder Case

Since the optimal buy-out strategy must be recursively derived, in order to solve for the optimal buy-out strategy for the 3-bidder case, we must first solve for that of the 2-bidder case. Even with a buy-out price, a basic result of the standard English auction remains true: The bidders will stay active in the auction as long as the standing price is lower than his valuation of the object. The complication comes from the fact that, at every

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5 For justification and consequence of this assumption, please see Chen et al. (2006).

6 The model can be easily extended to the case with reservation price.

7 The solution of the two-bidder case has been discussed in Chen et al. (2006). We include this discussion here for the sake of completeness.
prevailing price, now he has the additional option to pre-empt his opponent by buying
the objective immediately at the buy-out price $v_b$.

Given $v_b$, let $p_2(v)$ be the buy-out strategy of the bidder whose valuation of the objec-
tive is $v$.\footnote{The subscript 2 is to denote the 2-bidder case.} That is, a bidder who values the object at $v$ is willing to buy out the object (by paying $v_b$) when the prevailing price reaches $p_2(v)$. Since the greater the value of $v$, the
more willing is the bidder to obtain the object immediately by paying $v_b$, we know that
$p_2(v)$ is a decreasing function. It turns out to be easier to work with the inverse function
of $p_2(v)$. Let $v_2(p)$ be the inverse function of $p_2(v)$. It relates the prevailing price $p$ with
bidder’s valuation $v$, who at $p$ is just willing to buy out the objective. That is, a bidder
with valuation $v_2(p)$ is just willing to obtain the object by paying the buy-out price, when
the prevailing price reaches $p$. Similarly, if a bidder with valuation $v$ is willing to buy out
the object, when the prevailing price is $p$, then a bidder with valuation $v' > v$ will be even
more willing to do so at that moment. This implies that $v_2(p)$ is a decreasing function.

Suppose that both bidders are still active at the moment when the prevailing price
is $p$. This implies that the valuations of both bidders are greater than $p$, which in turn
implies that the possible valuations of any bidder must be distributed on $[p, \bar{v}]$. Moreover,
by definition of $v_2(p)$, any bidder with valuation $v > v_2(p)$ would have bought out the
object before the price has risen to $p$. The fact that this object has not been bought
out at price $p$ implies that the bidder’s possible valuations cannot lie in $(v_2(p), \bar{v}]$. As
a result, both bidders’ valuations must lie in $[p, v_2(p)]$. In other words, if both bidders
are still active when the prevailing price is $p$, then (by Bayes rule) any bidder’s possible
valuations of the object must be distributed uniformly on $[p, v_2(p)]$.\footnote{The subscript 2 is to denote the 2-bidder case.}
Consider the decision of a bidder (whose valuation is $v$) at the moment when the prevailing price is $p < v$. If he buys the object immediately with buy-out price $v_b$, his utility will be $u(v, v_b) = (v - v_b)^\alpha/\alpha$. If instead he holds out and waits until price is $p + dp$ to buy out the object, then he will face three possible outcomes. First, his opponent buys out the object while he waits. Second, his opponent drops out between $p$ and $p + dp$. Third, neither of the above happens so that he eventually buys out the object when the prevailing price is $p + dp$. Whether the bidder should buy the object immediately (by paying $v_b$), or waits until $p + dp$, depends on the difference of the utility between an immediate buy-out and the combined expected utility under the three possible outcomes of waiting until $p + dp$.

Figure 1 depicts the possible intervals at which outcomes 1 and 2 occur. When the valuation of the bidder’s opponent lies in $[v_2(p) + dv_2(p), v_2(p)]$, then his opponent will buy out the object while he waits. This is the first outcome we mentioned above, which occurs with probability $-\frac{dv_2(p)}{v-p}$, and his utility is 0. Similarly, if his opponent’s valuation lies in $[p, p + dp]$, then his opponent will drop out while he waits, and he will win the bidding with price $p$. This is the second outcome mentioned, which occurs with probability $\frac{dp}{v-p}$, and his utility is $(v - p)^\alpha/\alpha$. Under the third outcome, which occurs with probability $1 - \left(-\frac{dv_2(p)}{v-p} + \frac{dp}{v-p}\right)$, his utility is $(v - v_b)^\alpha/\alpha$.

\textsuperscript{9} Since $dv(p) = v'(p)dp$ and $v'(p) < 0$, $dv(p) < 0$. 
outcome 2 (opponent drops out),
if opponent’s valuation lies here

outcome 1 (opponent buys out),
if opponent’s valuation lies here

\[ p \quad p + dp \quad v_2(p) + dv_2(p) \quad v_2(p) \]

Figure 1: Possible outcomes of waiting.

The total expected utility of waiting until \( p + dp \) to buy out is thus

\[
\frac{dp}{v - p} \frac{(v - p)^{\alpha}}{\alpha} + \left[ 1 + \frac{dv_2(p)}{v - p} - \frac{dp}{v - p} \right] \frac{(v - v_b)^{\alpha}}{\alpha}.
\]  

(1)

The total change in utility of waiting until \( p + dp \) to buy out, instead of buying out now, is thus

\[
du = \frac{dp}{v - p} \frac{(v - p)^{\alpha}}{\alpha} + \frac{dv(p) - dp \,(v - v_b)^{\alpha}}{(v - p)}\alpha.
\]

(2)

For the function \( v_2(p) \) to be the optimal buy-out strategy, it must be the case that \( \frac{du}{dp} = 0 \), i.e., the first-order condition must hold at every \( p \). This implies that

\[
(v - p)^{\alpha} - (v - v_b)^{\alpha} = -(v - v_b)^{\alpha} \frac{dv_2}{dp}.
\]

(3)

Let \( y = v - v_b \) and \( x = v_b - p \), then \( \frac{dv}{dp} = -\frac{dy}{dx} \), and equation (3) becomes

\[
(x + y)^{\alpha} - y^{\alpha} = y^{\alpha} \frac{dy}{dx}.
\]

(4)

Since \( v_2(v_b) = v_b \), the solution of (4) must pass through \((x, y) = (0, 0)\). It is difficult to directly solve for equation (4), but this boundary condition and the fact that (4) is homogeneous of degree \( \alpha \) on both sides suggest that the solution is linear. We thus let \( x = \mu y \), then (4) becomes

\[
(1 + \mu)^{\alpha} = 1 + \frac{1}{\mu}.
\]

(5)

\(^{10}\) If the current price is \( v_b \), and buy out price is \( v_b \), then it must be optimal to buy out immediately.
Denote $\mu_2$ as the solution of (5). It is easy to see that $\mu_2$ exists and is unique. Moreover, $\mu_2 \geq 1$ and that $\mu_2$ is decreasing in $\alpha$. We thus have $x = \mu_2 y$. Substituting for $y = v - v_b$ and $x = v_b - p$ we eventually have

$$v_2(p) = \left(1 + \frac{1}{\mu_2}\right)v_b - \frac{p}{\mu_2}.$$  

(6)

The function $v_2(p)$ is the explicit form of the inverse of a bidder’s optimal buy-out strategy. Solving for the inverse of the function $v_2(p)$ we have

$$p_2(v) = (1 + \mu_2)v_b - \mu_2 v.$$  

(7)

The function $p_2(v)$ is exactly the optimal buy-out strategy of the bidder. It shows that a bidder, whose valuation of the object is $v$, will be willing to buy out the object (by paying $v_b$) when the prevailing price reaches $(1 + \mu_2)v_b - \mu_2 v$. Given the optimal buy-out policy, the optimal strategy of the bidder with valuation $v$ is then easy to describe: Stay active as long as the prevailing price is lower than $p_2(v)$, and buy it out as soon as price reaches $p_2(v)$. Note that since a bidder will consider buying out only if $v > v_b$, we know $v - p_2(v) = (1 + \mu_2)(v - v_b) > 0$. That is, if a bidder will buy out the object, then he will do so before the price reaches his valuation. This also implies that the transaction price cannot be higher than $v_b$. In other words, by setting $v_b$ as the buy-out price, the seller essentially sets $v_b$ as the upper-bound for possible transaction prices.

For the 2-bidder case, the optimal strategies derived in Hidvegi et al. (2006) and Reynolds and Wooders (2006) are also subgame perfect, as the game will end whenever one bidder drops out. When controlled for differences in the assumptions on distribution and utility functions, the differential equations in their papers (Theorem 1 in the former; Proposition 3 in the latter) will be the same as equation (3).
The expected utility of a bidder having valuation $v$ (when the prevailing price is $p < v_b$) under the optimal buy-out strategy, $p_2(v)$, is thus

$$U_2(v, p) = \frac{v_2(p) - v}{v_2(p) - p} \cdot 0 + \frac{v - p_2(v)}{v_2(p) - p} \frac{(v - v_b)^\alpha}{\alpha} + \frac{1}{v_2(p) - p} \int_{p}^{p_2(v)} \frac{(v - x)^\alpha}{\alpha} dx; \quad (8)$$

where the first term is the bidder’s expected utility when his opponent buys out before him; the second term is his expected utility when his opponent’s valuation is greater than $p_2(v)$ (but smaller than $v$) so that he wins by buy-out; the third term is his expected utility when his opponent drops out before price reaches $p_2(v)$, in that case he wins the item by paying the opponent’s valuation. Straightforward calculation shows that

$$U_2(v, p) = \frac{1}{\alpha(v_b - p)} \left[ \frac{\mu_2}{(1 + \alpha)(1 + \mu_2)}(v - p)^{1+\alpha} \right. \right.$$

$$+ \left. \frac{\mu_2(1 + \alpha) - \mu_2(1 + \mu_2)^\alpha}{1 + \alpha}(v - v_b)^{1+\alpha} \right]. \quad (9)$$

With this we can now proceed to the 3-bidder case. Let $p_3(v)$ be the buy-out strategy, and $v_3(p)$ its inverse. If instead of following $p_3(v)$ the bidder delays and waits until $p_3(v) + dp$ to buy out, then his potential gain is that one of his opponents might drop out during this delay. In that case his expected utility is $U_2(v, p)$. The case that both opponents drop out is of second-order, and can be ignored. Therefore his net gain of delay to buy out is

$$U_2(v, p) - \frac{(v - v_b)^\alpha}{\alpha}. \quad (10)$$

Since this occurs with probability

$$\frac{2(v_3(p) - p)dp}{(v_3(p) - p)^2}, \quad (11)$$

his expected net gain is

$$\frac{2dp}{v_3(p) - p} \left[ U_2(v, p) - \frac{(v - v_b)^\alpha}{\alpha} \right]. \quad (12)$$
On the other hand, the potential loss of delay is that one of his opponents buys out the object during \( p \) and \( p + dp \), in that case his loss is \( (v - v_b)^\alpha / \alpha \).\(^{11}\) Since this occurs with probability \(-2(v_3(p) - p)dv_3/(v_3(p) - p)^2\), the expected loss of delay is \[
-\frac{2dv_3}{v_3(p) - p} \frac{(v - v_b)^\alpha}{\alpha}.
\] (13)

That \( p_3(v) \) is optimal means that the expected gain equals expected loss, i.e., first-order condition for \( p_3(v) \) must hold at every \( p \). Consequently, by plugging in the value of \( v_2(v, p) \) and making equation (12) equal to (13), we have the following differential equation:

\[
A \frac{v - v_b}{v_b - p} + B \frac{(v - p)^{1+\alpha}}{(v_b - p)(v - v_b)^\alpha} - 1 + \frac{1}{p'_3(v)} = 0;
\] (14)

where \( A = \mu_2(1 - \frac{(1+\mu_2)^\alpha}{1+\alpha}) \) and \( B = \frac{\mu_2}{(1+\alpha)(1+\mu_2)} \).

Let \( y = v - v_b \) and \( x = v_b - p \), then \( dy = dv \) and \( dx = -dp \). As a result, \( \frac{dy}{dx} = -\frac{dv_3(p)}{dp} \).

Equation (14) then becomes:

\[
A \frac{y}{x} + B \frac{(x + y)^{1+\alpha}}{xy^\alpha} - 1 = \frac{dy}{dx}.
\] (15)

This equation is homogeneous of degree 0 in \( x \) and \( y \). Moreover, it passes through the point \((x, y) = (0, 0)\). Again, a natural conjecture for the solution (15) is that \( x = \mu y \), where \( \mu \) is a constant. In that case equation (15) becomes

\[
B(1 + \mu)^\alpha = 1 - \frac{A}{1 + \mu}.
\] (16)

First note that \( A = \mu_2(1 - \frac{(1+\mu_2)^\alpha}{1+\alpha}) = \mu_2(1 - \frac{1}{1+\mu_2}) \) by the definition of \( \mu_2 \). We have shown in Chen et al. (2006) that \( \frac{1}{\mu_2} > \alpha \). As a result, \( A < 0 \). Moreover, it can be easily seen that \( B < 1 \). Equation (16) is thus plotted in Figure 2. Obviously, there exists a unique

\(^{11}\) Similarly, the probability that both opponents buy out is of second-order, and can be ignored.
solution of for (16), $\mu_3$, and it is in inverse relation with $\alpha$. The solution for (15) is thus $x = \mu_3 y$. Plugging $x = v - v_b$ and $y = v_b - p$ into this equation, we have

$$p_3(v) = (1 + \mu_3)v_b - \mu_3 v.$$  \hfill (17)

![Figure 2. Determination of the Value of $\mu_3$.](image)

Equation (17) is exactly the same as equation (6), except that the values of $\mu_2$ and $\mu_3$ are different. In fact, we can further show that $\mu_3 > \mu_2$. For this we only have to prove that

$$1 - \frac{A}{1 + \mu_2} - B(1 + \mu_2)\alpha > 0.$$  \hfill (18)

Plugging the values of $A$ and $B$ into the left-hand-side of (18), we have

$$1 - \frac{A}{1 + \mu_2} - B(1 + \mu_2)\alpha = 1 - \frac{(1 + \alpha)\mu_2 - \mu_2(1 + \mu_2)\alpha}{(1 + \alpha)(1 + \mu_2)} = 1 - \frac{\mu_2}{(1 + \mu_2)} > 0.$$  \hfill (19)
The fact that $\mu_3 > \mu_2$ implies that the threshold price to buy out is different in the 2-bidder and the 3-bidder cases. More importantly, since both $p_3(v)$ and $p_2(v)$ are defined over $[v_b, \bar{v}]$, and that $p_3(v_b) = p_2(v_b)$, the threshold price to buy out for a bidder is lower in the 3-bidder case than in the 2-bidder case (i.e., $p_3(v) < p_2(v)$ for all $v \in [v_b, \bar{v}]$). This in turn implies that when one of the bidders drops out, the remaining bidders will respond by raising their threshold auction prices at which they are willing buy out. Put differently, when some bidder drops out, the remaining bidders will wait until the prevailing price is higher than they had planned (in the 3-bidder case) in order to buy out.

Define $v^c_3$ so that $p_3(v^c_3) = 0$. Any bidder whose valuation of the item is greater or equal to $v^c_3$ will be willing to buy out the item at the beginning of the auction (i.e., when the prevailing price is 0). With our assumption that the bidder who has the highest valuation of the item will win in buy-out, the dynamics of the whole bidding process is now easy to describe. For convenience of explanation, and without loss of generality, assume that $v^1 > v^2 > v^3$. The bidding starts with price 0, and every bidder has a threshold price to buy out as a function of his own valuation, namely $p_i(v)$, $i = 1, 2, 3$. If there is one (or more than one) bidder with valuation greater than $v^c_3$, then the item is sold with $v_b$ immediately (to the bidder who has the highest valuation, by our assumption). If not, then the price rises continuously.

If $v^3 \geq p_3(v^1)$, then bidder 3 has not yet dropped out when price rises to the level at which bidder 3 will drop out. In that case the item is sold to bidder 1 with the buy-out price $v_b$. If $v^3 < p_3(v^1)$, then bidder 3 will drop out before any other buys out the item. When this happens, bidders 1 and 2 will raise their threshold prices from $p_3(v^2)$ and $p_3(v^1)$

\[ v^c_3 = (1 + \frac{1}{p_3(v_b)}). \]
to \( p_2(v^2) \) and \( p_2(v^1) \), respectively. Again, if \( v^2 \geq p_2(v^1) \), then bidder 1 will eventually win the item by paying the buy-out price \( v_b \). If \( v^2 < p_2(v^1) \), then bidder 1 wins by paying bidder 2’s valuation, \( v^2 \).

The expected utility of a bidder with valuation \( v \), when prevailing price is \( p \), is thus

\[
U_3(v, p) = \frac{(v_3(p) - v)(v_3(p) - p + v - p_3(v))}{(v_3(p) - p)^2} \cdot 0
\]

\[
+ \frac{(v - p_3(v))^2}{(v_3(p) - p)^2} \cdot \frac{(v - v_b)^\alpha}{\alpha} + \frac{2}{(v_3(p) - p)^2} \int_p^{p_3(v)} \int_{v(1)}^{v_3(v(1))} U_2(v, v(1)) \, dv(2) \, dv(1)
\]

\[
= \frac{1}{\alpha(v_b - p)^2} \left[ \frac{2 \mu_2 \mu_3(v - p)^{2+\alpha}}{(1 + \alpha)(2 + \alpha)(1 + \mu_2)(1 + \mu_3)} + \frac{2 \mu_3^2(\alpha \mu_2 - 1)}{(1 + \alpha)(2 + \alpha)(1 + \mu_2)} + \frac{2 \mu_2^2(\alpha \mu_3 - 1)}{(1 + \alpha)(1 + \mu_3)} \right] (v - v_b)^{2+\alpha}
\]

\[
- \frac{2(\alpha \mu_2 - 1) \mu_3}{(1 + \alpha)(1 + \mu_3)} (v - v_b)^{1+\alpha} (v_b - p)
\]

(20)

where \( v(i) \) is the order statistic for a bidder with the \( i \)-th lowest valuation in the 2-bidder case.

### 2.2 The General Case

In the general case when there are \( n \) bidders, let \( p_n(v) \) be the buy-out strategy of the bidders, and \( v_n(p) \) its inverse function. Under the buy-out strategy \( p_n(v) \), let \( U_n(v, p) \) be the expected utility of a bidder whose valuation of the item is \( v \), calculated at the prevailing price \( p \).

There is a recursive relation between the “value functions” \( U_n \) and \( U_{n-1} \). Specifically, consider a bidder whose valuation is \( v \). Then if the prevailing price is \( p \), there are three components for \( U_n(v, p) \) (see Figure 3). In the figure, \( v_{(n-1)} \) and \( v_{(1)} \) denote the order
statistic of the maximum and minimum valuation among $n - 1$ bidders, respectively. Since $v_{(n-1)}$ is always greater than $v_{(1)}$, we only have to consider the area above the 45° line.

In region $E_1$, since $v_{(n-1)} > v$ and $p_n(v_{(1)}) > v_{(1)}$, his opponent who has the highest valuation will win before the one having the lowest valuation drops out. In this case his utility is 0. In region $E_2$, since $v > v_{(n-1)}$ and $p_n(v) < v_{(1)}$, the bidder will buy out the item before any of his opponents drop out. In this case his utility is $(v - v_b)\alpha/\alpha$. In region $E_3$, we have $v_{(1)} < p_n(v)$ and $v_{(n-1)} < v_n(v_{(n-1)})$. In this case, the lowest-valuation bidder (having valuation $v_{(1)}$) will leave before anyone buys out. Consequently, his utility is the expected utility of the $(n - 1)$-bidder game, evaluated at the price when the lowest-valuation drops out (which is $v_{(1)}$), $U_{n-1}(v, v_{(1)})$. Thus $U_n(v, p)$ can be written recursively.
as

\[ U_n(v, p) = \text{Prob}(E_1) \cdot 0 + \text{Prob}(E_2) \frac{(v - v_b)^\alpha}{\alpha} \]

\[ + \int_p^{p_n(v)} \int_{v(1)}^{v_n(v(1))} \frac{(n - 1)(n - 2)(v_{n-1} - v(1))^{n-3}}{(v_n(p) - p)^{n-1}} U_{n-1}(v, v(1)) dv_{n-1} dv(1) \]

\[ = \left( \frac{v - p_n(v)}{v_n(p) - p} \right)^{n-1} \frac{(v - v_b)^\alpha}{\alpha} \]

\[ + \int_p^{p_n(v)} \frac{(n - 1)(v_n(v(1)) - v(1))^{n-2}}{(v_n(p) - p)^{n-1}} U_{n-1}(p, v(1)) dv(1). \] (21)

We can now derive the differential equation characterizing the optimal buy-out strategy. Suppose the bidder delays his buy-out from \( p_n(v) \) to \( p_n(v) + dp \). Then, similar to the reasoning of the 3-bidder case, his expected gain is

\[ \frac{(n - 1)(v_n(p) - p)^{n-1} dp}{(v_n(p) - p)^{n-1}} \left[ U_{n-1}(v, p) - \frac{(v - v_b)^\alpha}{\alpha} \right]. \] (22)

His expected loss is

\[ -\frac{(n - 1)(v_n(p) - p)^{n-1} dv_n}{(v_n(p) - p)^{n-1}} \frac{(v - v_n)^\alpha}{\alpha}. \] (23)

That \( p_n(v) \) is the optimal means that the sum of gain and loss is 0, i.e.,

\[ U_{n-1}(v, p) - \frac{(v - v_b)^\alpha}{\alpha} = -\frac{dv_n}{dp} \frac{(v - v_b)^\alpha}{\alpha}, \] (24)

or

\[ \frac{U_{n-1}(v, p)}{(v - v_b)^\alpha} - 1 + \frac{1}{p'_n(v)} = 0. \] (25)

This is the differential equation for the optimal buy-out strategy \( p_n(v) \). It is instructive to compare our differential equation (25) with that in Theorem 1 of Hidvegi et. al. (2006) which, under the assumptions we impose, reduces to

\[ \frac{(v - p_n(v))^\alpha}{(v - v_b)^\alpha} - 1 + \frac{1}{p'_n(v)} = 0. \] (26)
The only difference between (25) and (26) is the numerator of the first term. In Hidvegi et al. (2006), since a bidder cannot observe drop-out while he delays buy-out, it is only if all his opponents leave before price reaches his buy-out price $p_n(v)$, that the uncertainty is resolved and he is sure of winning the bidding, which gives him a utility of $\frac{(v - p_n(v))^\alpha}{\alpha}$. In our model, however, since the bidder is aware when certain bidder drops out, the benefit of delay is the expected payoff after one of the bidders leaves, $U_{n-1}(v, p)$.\(^{13}\)

We can now state our main theorem.

**Theorem.** For any $n \geq 2$, there exists, for the $n$-bidder auction, a unique perfect Bayesian buy-out equilibrium $\{p_i(v) = (1 + \mu_i)v_b - \mu_i v\}_{i=2}^n$; where $\mu_i$’s are positive constants such that $\mu_{i+1} > \mu_i$ for all $i = 2, \cdots, n - 1$. When the prevailing price is $p$ and when there are still $i$ ($n \geq i \geq 2$) active bidders, a bidder with valuation $v > p$ will buy out the item as soon as price reaches $p_i(v)$.

In order to prove the theorem, we use induction on the number of bidders, $n$. We separate the proof into three lemmas.

**Lemma 1.** If in the $n$-bidder case the optimal buy-out strategy is linear in the form of $p_n(v) = (1 + \mu_n)v_b - \mu_n v$, and that $U_n(v, p)$ is homogeneous of degree $\alpha$ in $(v - v_b)$, $(v_b - p)$ and $(v - p)$ of the form of equation (27), then the same must also be true for $U_{n+1}(v, p)$ in the $(n + 1)$-bidder case.

$$U_n(v, p) = \frac{1}{\alpha(v_b - p)^{n-1}} \left\{ c_n(\mu_2, \cdots, \mu_n)(v - p)^{n-1+\alpha} \right.$$  

$$\quad + \sum_{i=0}^{n-2} c_{n,i}(\mu_2, \cdots, \mu_n)(v - v_b)^{n-1-i+\alpha}(v_b - p)^{i} \right\}; \quad (27)$$

\(^{13}\) As is explained earlier, multi-bidder drop-out between $p$ and $p + dv$ is of second-order.
where \( c_n(\cdot) \) and \( c_{n,i}(\cdot) \) are all functions of \( \mu_2, \cdots, \mu_n \), for \( i = 0, \cdots, n - 2 \).

Note that both \( U_3(v, p) \) and \( U_2(v, p) \) are in the form of (27).

Proof. Let \( y = v - v_b \), \( x = v_b - p \). Then \( v - p = x + y \), \( dv = dy \) and \( dp = -dx \). By our previous argument, the differential equation for the optimal buy-out strategy in the \((n + 1)\)-bidder case is

\[
\frac{U_n(v, p)}{(v - v_b)^\alpha} - 1 + \frac{1}{p'_{n+1}(v)} = 0. \tag{28}
\]

Plugging (27) into (28), and substitutive for \( x \) and \( y \), we have

\[
\frac{1}{x^{n-2}y^{\alpha}} \left\{ c_n(\mu_2, \cdots, \mu_n)(x+y)^{n-1+\alpha} + \sum_{i=0}^{n-2} c_{n,i}(\mu_2, \cdots, \mu_n)y^{n-1-i+\alpha}x^i \right\} - 1 = dy/dx. \tag{29}
\]

This is a homogeneous differential equation which passes through \((x, y) = (0, 0)\). Again conjecturing that \( x = \mu y \) and substitute it into (29), we have an equation for \( \mu \):

\[
\frac{1}{\mu^{n-1}} \left\{ c_n(\mu_2, \cdots, \mu_n)(\mu+1)^{n-1+\alpha} + \sum_{i=0}^{n-2} c_{n,i}(\mu_2, \cdots, \mu_n)\mu^i \right\} - 1 = \frac{1}{\mu}. \tag{30}
\]

Let the solution be \( \mu_{n+1} \). (For existence and uniqueness of \( \mu_{n+1} \), see Lemmas 2 and 3.) We thus have

\[
p_{n+1}(v) = (1 + \mu_{n+1})v_b - \mu_{n+1}v. \tag{31}
\]

Given that \( p_{n+1}(v) \) is a linear function, we can now show that \( U_{n+1}(v, p) \) is homogeneous of degree \( \alpha \) in \( (v - v_b), (v_b - p) \) and \( (v - p) \), and is in the form of (27):
Plugging (30) and (31) into (21), and after laborious calculation, we have

\[
U_{n+1}(v, p) = \frac{1}{\alpha(v_b - p)^n} \left[ c_{n+1}(\mu_2, \ldots, \mu_{n+1})(v - p)^{n+\alpha}
\right.
\]

\[
+ \sum_{i=0}^{n-1} c_{n+1,i}(\mu_2, \ldots, \mu_{n+1})(v - v_b)^{n-i+\alpha}(v_b - p)^i \right] ;
\]

where

\[
c_{n+1}(\mu_2, \ldots, \mu_{n+1}) = \frac{n\mu_{n+1}^n c_{n-1}(\mu_2, \ldots, \mu_n)}{(n + \alpha)(1 + \mu_{n+1})},
\]

\[
c_{n+1,0}(\mu_2, \ldots, \mu_n) = \mu_{n+1}^n - \frac{n\mu_{n+1}^n c_{n-1}(\mu_2, \ldots, \mu_n)}{(n - 1 + \alpha)(1 + \mu_{n+1})^{n-1+\alpha}}
\]

\[
- \frac{n\mu_{n+1}}{1 + \mu_{n+1}} \sum_{i=0}^{n-2} c_{n,i}(\mu_2, \ldots, \mu_{n+1}) \mu_{n+1}^{i+1},
\]

\[
c_{n+1,i}(\mu_2, \ldots, \mu_{n+1}) = \frac{n\mu_{n+1}^{i+1}}{i(1 + \mu_{n+1})} c_{n,i-1}(\mu_2, \ldots, \mu_n), \ i \geq 1.
\]

Obviously, \(U_{n+1}(v, p), c_{n+1}, c_{n+1,0} \) and \(c_{n+1,i} (i \geq 1)\) are all in the forms we have claimed.

\[\blacksquare\]

**Lemma 2.** Let \(\{p_i(v) = (1 + \mu_i)v_b - \mu_i v\}_{i=2}^n\) be the equilibrium buy-out strategy in the \(n\)-bidder case. Then in the \((n+1)\)-bidder case, there exists \(\mu_{n+1} > \mu_n\) such that

\(\{p_{i+1}(v) = (1 + \mu_{i+1})v_b - \mu_{i+1}v\}_{i=1}^n\) is the optimal buy-out strategy.

**Proof.** Rewriting (30), the equation for the solution of \(\mu_{n+1}\) becomes

\[
\frac{1}{\mu^{n+1}} \left[ c_n(\mu_2, \ldots, \mu_n)(1 + \mu)^{n-1+\alpha} + \sum_{i=0}^{n-2} c_{n,i}(\mu_2, \ldots, \mu_n)\mu^i \right] = 1 + \frac{1}{\mu}.
\]
Define the left-hand-side of (36) as \( f_{n+1}(\mu) \). Then using the recursive relation for \( c_n \) and \( c_{n-1,i} \)'s, it can be rewritten as

\[
f_{n+1}(\mu) = \frac{(n-1)\mu_n}{1 + \mu_n} \frac{1}{\mu^{n-1}} \left\{ \frac{c_{n-1}(\mu_2, \cdots, \mu_{n-1})}{(n-1+\alpha)} \left( (1 + \mu)^{n-1+\alpha} - (1 + \mu_n)^{n-1+\alpha} \right) \right. \\
\left. + \sum_{i=0}^{n-3} \frac{1}{1+i} c_{n-1,i}(\mu_2, \cdots, \mu_{n-1})(\mu^{i+1} - \mu^i) \right\} + \frac{1}{\mu^{n-1}} \mu^{n-1}. \tag{37}
\]

It is easy to see that \( f_{n+1}(\mu_n) = 1 < 1 + \frac{1}{\mu_n} \). Since \( f_{n+1}(\infty) = \infty \), by continuity of \( f_{n+1}(\cdot) \) we know that there exists \( \mu_{n+1} \in (\mu_n, \infty) \) such that (36) holds, i.e., \( f_{n+1}(\mu_{n+1}) = 1 + \frac{1}{\mu_{n+1}} \).

Finally, we show the uniqueness of \( \mu_{n+1} \). First note that the right-hand side of (36) is decreasing in \( \mu \). Second, the buy-out strategy \( p_n(v) = (1 + \mu_n)v_b - \mu_nv \) passes through \((v_b, v_b)\) for all \( n \); that is, \( p_n(v) \) rotates around the point \((v_b, v_b)\) as the value of \( n \) changes. Third, the more bidders there are, the more competitive the auction is, and thus the more willing the bidders will be to buy out the item. In other words, the prevailing price at which a bidder is willing to buy out will be lower when there are more bidders, which in turn implies that \( \mu_n \) is increasing in \( n \). Finally, \( f_{n+1}(\mu_n) = 1 < 1 + \frac{1}{\mu_n} \). Together, these four facts imply that in order to show that \( \mu_{n+1} \) is unique, it suffices to show that the left-hand-side of (36), i.e., \( f_{n+1}(\mu) \), is increasing in \( \mu \) for all \( \mu \geq \mu_n \).

**Lemma 3.** \( f_{n+1}(\mu) \) is increasing in \( \mu \) for \( \mu \geq \mu_n \).
Proof. It is straightforward to show (using the definition of $f_{n+1}(\mu)$) that

\[ f'_{n+1}(\mu) = -\frac{n-1}{\mu} f_{n+1}(\mu) + \frac{1}{\mu^{n-1}} \left\{ (n-1+\alpha)c_n(\mu_2, \cdots, \mu_n)(1+\mu)^{n-2+\alpha} \\
+ \sum_{i=1}^{n-2} ic_{n,i}(\mu_2, \cdots, \mu_n)\mu^{i-1} \right\} \\
= -\frac{n-1}{\mu} f_{n+1}(\mu) + \left( \frac{(n-1)\mu_n}{1+\mu_n} \right) \frac{1}{\mu_{n-1}} \left\{ c_{n-1}(\mu_2, \cdots, \mu_{n-1})(1+\mu)^{n-2+\alpha} \\
+ \sum_{i=1}^{n-2} c_{n-1,i-1}(\mu_2, \cdots, \mu_{n-1})\mu^{i-1} \right\} \right) \\
\]

(38)

As a result, in order to show that $f'_{n+1}(\mu) > 0$ for $\mu \geq \mu_n$, it suffices to prove that

\[ \frac{\mu_n}{1+\mu_n} f_n(\mu) - f_{n+1}(\mu) > 0 \text{ when } \mu > \mu_n. \]

The proof is rather straightforward but computationally tedious, and we relegate it to the appendix.

Combining the results in Lemmas 1, 2 and 3 we thus complete proof of the theorem.

To summarize, we have shown in Section 2.2 that any optimal buy-out strategy must satisfy (25). Since $v_2(p, v)$ and $p_2(v)$ satisfy the assumptions of Lemma 1, using introduction on $n$ and the results in Lemmas 1 and 2 we know that there exists an optimal buy-out strategy $\{p_i(v) = (1+\mu_n)v_b-\mu_n v\}_{i=2}^n$ for all $n \geq 2$. Finally, Lemma 3 shows that the optimal buy-out strategy is unique.
3 Conclusion

This paper derives the equilibrium strategy of the bidder in an English auction with buy-out. Unlike the previous literature, we assume that during the auction the bidders can observe dropouts of their opponents. In this case, the bidders’ strategies will become substantially more complicated, because they need to update their information (and thus modify their strategies accordingly) whenever a bidder drops out of auction. Under the assumptions imposed on the density function of the bidders’ valuations and their utilities, we are able to characterize the equilibrium strategies in closed-form solution. The equilibrium strategy is shown to be a function of a bidder’s own valuation and the number of active bidders. Moreover, a bidder will respond to dropout of opponents by delaying his buy-out timing. The assumptions we need to impose in order to derive closed-form solution, however, are relatively stringent. In particular, we assume that the bidders’ valuations are uniformly distributed, and that both the bidders and the sellers exhibit constant degree of risk-aversion. It will be of theoretical interest to see how our results can extend to a more general context.
Appendix: Proof that $\frac{\mu_n}{1+\mu_n} f_n(\mu) - f_{n+1}(\mu) > 0$ when $\mu > \mu_n$

We will first express $\frac{\mu_n}{1+\mu_n} f_n(\mu)$ and $f_{n+1}(\mu)$ in their Taylor expansions (around the value $\mu_n$). We show they have the same residual terms and thus cancel out in the subtraction. Then using induction on $n$ we prove that $\frac{\mu_n}{1+\mu_n} f_n(\mu)$ is greater than $f_{n+1}(\mu)$ for all $\mu > \mu_n$. First, omitting the arguments of the $c_i$ and $c_{i,j}$ functions, we can write the following recurrent formulas:

\begin{align*}
c_{n-1} &= \frac{n-2}{n-2+\alpha} \left( \frac{\mu_{n-1}}{1+\mu_{n-1}} \right) c_{n-2} = \cdots \\
&= \frac{(n-2) \cdots (n-i)}{(n-2+\alpha) \cdots (n-i+\alpha)} \left( \prod_{k=n-i+1}^{n-1} \frac{\mu_k}{1+\mu_k} \right) c_{n-i}, \ i = 1, \cdots, n-1; \quad (39) \\
c_{n-1,j} &= \frac{n-2}{j} \left( \frac{\mu_{n-1}}{1+\mu_{n-1}} \right) c_{n-2,j-1} = \cdots \\
&= \frac{(n-2) \cdots (n-i)}{j \cdots (j-i+2)} \left( \prod_{k=n-i+1}^{n-1} \frac{\mu_k}{1+\mu_k} \right) c_{n-i,j-i+1}, \ i = 1, \cdots, j+1, \quad (40)
\end{align*}

for all $j = 1, \cdots, n-3$.

We need the following Taylor’s expansions:

\begin{align*}
(1 + \mu)^{n-2+\alpha} &\quad = \quad (1 + \mu_n)^{n-2+\alpha} + \sum_{i=1}^{n-2} \frac{(n-2+\alpha) \cdots (n-1-i+\alpha)(1 + \mu_n)^{n-2-i+\alpha}}{i!} (\mu - \mu_n)^i \\
&\quad + \frac{(n-1+\alpha) \cdots (1+\alpha)\alpha(1 + \bar{\mu}_n)^{\alpha-1}}{(n-1)!} (\mu - \mu_n)^{n-1}.
\end{align*}

(41)
\[(1 + \mu)^{n-1+\alpha} = (1 + \mu)(1 + \mu)^{n-1+\alpha}\]
\[(1 + \mu_n)^{n-1+\alpha} + \sum_{i=1}^{n-2} \frac{(n - i + \alpha)}{i!} (1 + \mu_n)^{n-1-i+\alpha} \mu - \mu_n)^i\]
\[\frac{(n - 2 + \alpha) \cdots (1 + \alpha)(1 + \mu_n)^{\alpha}}{(n-1)!} \left[ n - 1 + \alpha \left( \frac{1 + \mu_n}{1 + \mu_n} \right)^{\alpha-1} \right] (\mu - \mu_n)^{n-1}\]
\[\frac{(n - 1 + \alpha) \cdots (1 + \alpha) \alpha(1 + \mu_n)^{\alpha-1}}{(n-1)!} (\mu - \mu_n)^n. \quad \text{(42)}\]
\[\mu^j = \mu^j_n + \sum_{i=1}^{n-2} \frac{j(j-1) \cdots (j-i+1)}{i!} \mu^j_n - \mu_n)^i. \quad \text{(43)}\]
\[\mu^j_n = \mu^j_n \mu^{n-2+\alpha}\]
\[\mu^j_n = \mu^j_n + \sum_{i=1}^{n-2} \frac{(n - i + \alpha)}{i!} [i + (n - 1 + \alpha) \mu_n](1 + \mu_n)^{n-2-i+\alpha} (\mu - \mu_n)^i\]
\[\frac{(n - 2 + \alpha) \cdots (1 + \alpha)}{(n-1)!} [(n - 1)(1 + \mu_n)^{\alpha} + (n - 1 + \alpha) \alpha \mu_n (1 + \mu_n)^{\alpha-1}](\mu - \mu_n)^{n-1}\]
\[\frac{(n - 1 + \alpha) \cdots (1 + \alpha) \alpha(1 + \mu_n)^{\alpha-1}}{(n-1)!} (\mu - \mu_n)^n. \quad \text{(44)}\]

In (41), (42) and (44), \(\bar{\mu}_n \in (\mu_n, \mu)\).

Using equations (39)-(44), we have
\[f_{n+1}(\mu) = \frac{1}{\mu^{n-1}} \mu^{n-1} + \left( \frac{(n-1)\mu_n}{1 + \mu_n} \right) \frac{1}{\mu^{n-1}} \left\{ \frac{c_{n-1}}{n - 1 + \alpha} [(1 + \mu)^{n-1+\alpha} - (1 + \mu_n)^{n-1+\alpha}] + \sum_{j=1}^{n-2} \frac{1}{j} c_{n-1-j-1} (\mu^j - \mu_n^j) \right\}\]
\[
\begin{align*}
\frac{1}{\mu^{n-1}} + \frac{(n-1)\mu_n}{1+\mu_n} \frac{1}{\mu^{n-1}} \{ \\
\sum_{i=1}^{n-2} \frac{(n-2+i-\alpha) \cdots (n-i+\alpha)(1+\mu_n)^{n-i-1+\alpha}}{i!} (\mu - \mu_n) \\
+ \frac{(n-1+\alpha) \cdots (1+\alpha)\alpha(1+\hat{\mu}_n)^{\alpha-1}(1+\mu_n)}{(n-1)!} (\mu - \mu_n)^n \\
+ \frac{(n-1+\alpha) \cdots (1+\alpha)\alpha(1+\hat{\mu}_n)^{\alpha-1}}{(n-1)!} (\mu - \mu_n)^n \} \\
+ \sum_{j=1}^{n-2} c_{n-1,j-1} \sum_{i=1}^{j} (j-1) \cdots (j-i+1)\mu_n^{j-1} (\mu - \mu_n)^i \\
= \frac{\mu_n}{\mu^{n-1}} + \frac{(n-1)\mu_n}{1+\mu_n} \frac{1}{\mu^{n-1}} \{ \\
\sum_{i=1}^{n-2} \left[ (n-2) \cdots (n-i) \left( \prod_{k=n-i+1}^{n-1} \frac{\mu_k}{1+\mu_k} \right) \mu_n^{n-i-1} f_{n-i+1}(\mu_n) \right] \frac{(\mu - \mu_n)^i}{i!} \\
+ \left( \prod_{k=2}^{n-1} \frac{\mu_k}{1+\mu_k} \right) f_2(\mu_n) \frac{n-1+\alpha}{n-1+\alpha} (\mu - \mu_n)^n \\
+ \left( \prod_{k=2}^{n-1} \frac{\mu_k}{1+\mu_k} \right) \frac{1}{n-1} \frac{\alpha(1+\hat{\mu}_n)^{\alpha-1}}{n-1+\alpha} (\mu - \mu_n)^n \} \}. \tag{45}
\end{align*}
\]

Similarly,\(^{14}\)

\[
\begin{align*}
f_n(\mu) \\
= \frac{1}{\mu^{n-2}} \left\{ c_{n-1} (1+\mu)^{n-2+\alpha} + \sum_{j=0}^{n-3} c_{n-1,j} \mu^j \right\} \\
= \frac{1}{\mu^{n-1}} \left\{ c_{n-1} [\mu(1+\mu)^{n-2+\alpha} - \mu_n(1+\mu_n)^{n-2+\alpha}] + \sum_{j=1}^{n-2} c_{n-1,j-1} \mu^j - \mu_n^j \right\} + \left( \frac{1+\mu_n}{\mu_n} \right) \frac{\mu_n^{n-1}}{\mu^{n-1}}
\end{align*}
\]

\(^{14}\) Our convention is that whenever \(i > j\), then \(\sum_{n=1}^{j} a_n = 0\).
\[
\begin{align*}
&= \frac{1}{\mu^n} \left\{ c_{n-1} \sum_{i=1}^{n-2} \frac{(n-2+\alpha) \cdots (n-i+\alpha)}{i!} (i+\alpha) \mu_n (1+\mu_n)^{n-2-i} \mu^n \right. \\
&\quad + \frac{(n-2+\alpha) \cdots (1+\alpha)}{(n-1)!} \left[ (n-1) (1+\mu_n)^\alpha + (n-1+\alpha) \alpha \mu_n (1+\mu_n)^{\alpha-1} \right] \mu^n \\
&\quad + \frac{(n-1+\alpha) \cdots (1+\alpha) \alpha (1+\mu_n)^{\alpha-1}}{(n-1)!} \mu^n \right\} \\
&\quad + \frac{\mu^{n-1}}{\mu^n} \left\{ \sum_{j=1}^{n-2} \sum_{i=1}^{n-2} \frac{j(j-1) \cdots (j-i+1)}{i!} \mu_{n-i}^{j-i} \right. \\
&\quad \left. + \frac{\mu^{n-i}}{1+\mu_{n-i}} (n-i-1) f_{n-i}(\mu_n) \right\} \frac{(\mu-\mu_n)^i}{i!} \\
&\quad + \left( \prod_{k=2}^{n-1} \frac{\mu_k}{1+\mu_k} \right) \left[ (n-1) f_2(\mu_n) + (n-1+\alpha) \alpha \mu_n (1+\mu_n)^{\alpha-1} \right] \frac{(\mu-\mu_n)^{n-1}}{n-1} \\
&\quad + \left( \prod_{k=2}^{n-1} \frac{\mu_k}{1+\mu_k} \right) \left( \frac{n-1+\alpha}{n-1} \alpha (1+\mu_n)^{\alpha-1} \right) \frac{(\mu-\mu_n)^n}{n-1} \\
&\quad + \frac{\mu^{n-1}}{\mu^n} \left( \frac{1+\mu_n}{\mu_n} \right) \frac{\mu_{n-1}^{n-1}}{\mu_{n-1}^{n-1}}.
\end{align*}
\]
Again we will prove our result by using induction in \( n \). As a starting point for our induction on \( n \), we need to show that \( f_3(\mu) \) is increasing for all \( \mu > \mu_3 \). Using the definition in (36) and the facts that \( c_2 = \mu_2/(1+\alpha)(1+\mu_2) \) and \( c_{2,0} = \mu_2 - \mu_2(1+\mu_2)^\alpha/(1+\alpha) \), we know that
\[
\begin{align*}
f_3'(\mu) &= \mu^{-2} \left[ \frac{\mu \mu_2}{1+\mu_2} (1+\mu)^\alpha - \frac{\mu_2}{1+\mu_2} (1+\mu)^{1+\alpha} - \mu_2 + \frac{\mu_2(1+\mu_2)^\alpha}{1+\alpha} \right] \\
&= \mu^{-2} \left[ \frac{\mu_2}{1+\mu_2} (1+\mu)^\alpha \frac{\mu_2}{1+\alpha} - \mu_2 + \frac{1+\mu_2}{1+\alpha} (1+\mu_2)^\alpha \right] \\
&= \mu^{-2} \mu_2 \left[ \frac{(1+\mu)^\alpha}{1+\alpha} \mu_2 - \frac{1+\mu_2}{1+\alpha} (1+\mu_2)^\alpha \right] \\
&= \mu^{-2} \mu_2 \left[ \frac{(1+\mu)^\alpha}{1+\alpha} \mu_2 - \frac{1+\mu_2}{1+\alpha} \right] \\
&= \mu^{-2} \mu_2 \left[ \frac{(1+\mu_2)^\alpha}{1+\alpha} \mu_2 - \frac{1+\mu_2}{1+\alpha} \right] \\
&= \mu^{-2} \mu_2 \left[ \frac{(1+\mu)^\alpha}{1+\alpha} \mu_2 - \frac{1+\mu_2}{1+\alpha} \right];
\end{align*}
\]
where the last equality comes from the fact that \( (1+\mu_2)^\alpha = 1+\frac{1}{\mu_2} \).
For all $\mu > \mu_2$, (47) is greater than
\[
\frac{\mu^{-2}\mu_2}{1 + \alpha} \left[ \frac{(1 + \mu_2)^\alpha}{1 + \mu_2} (\mu\alpha_2 - 1) + \left( \frac{1}{\mu_2} - \alpha \right) \right].
\] (48)

Again, using the fact that $(1 + \mu_2)^\alpha = 1 + \frac{1}{\mu_2}$, (48) becomes
\[
\frac{\mu^{-2}\mu_2}{1 + \alpha} \left( \mu_2 - \frac{1}{\mu_2} (1 - \mu_2\alpha) \right) = \frac{\mu^{-2}\mu_2}{1 + \alpha} \frac{\mu_2 - \frac{1}{\mu_2}}{\mu_2(1 + \mu_2)} (1 - \mu_2\alpha) > 0.
\] (49)

The last inequality comes from the fact that $\mu_2 \geq 1$ and $\frac{1}{\mu_2} > \alpha$. We thus know that $f_3'(\mu) > 0$ for all $\mu > \mu_2$.

Suppose our claim is true for all $k \leq n$. Then from (45) and (46), we obtain

\[
f_{n+1}'(\mu) = \left( \frac{\mu_n}{1 + \mu_n} \right) f_n(\mu) - f_{n+1}(\mu)
\]
\[
= \left( \frac{\mu_n}{1 + \mu_n} \right) \frac{1}{\mu^{n-1}} \left\{ \sum_{i=1}^{n-2} (n - 2) \cdots (n - i - 1) \left( \prod_{k=n-i+1}^{n-1} \frac{\mu_k}{1 + \mu_k} \right) \mu_{n-i} \right\}
\]
\[
+ \left( \frac{\mu_{n-i}}{1 + \mu_{n-i}} \right) \left[ \frac{\alpha (1 - \left( \frac{1+\bar{\mu}_n}{1+\mu_n} \right)^{\alpha-1})}{(n-1)\alpha(1+\bar{\mu}_n)^{\alpha-1} - 1} \sum_{i=1}^{n-2} (n - 2) \cdots (n - i - 1) \left( \prod_{k=n-i+1}^{n-1} \frac{\mu_k}{1 + \mu_k} \right) \mu_{n-i} \right]
\]
\[
+ \left( \frac{\prod_{k=1}^{n-1} \mu_k}{1 + \mu_k} \right) \left[ \frac{\alpha (1 - \left( \frac{1+\bar{\mu}_n}{1+\mu_n} \right)^{\alpha-1})}{(n-1)\alpha(1+\bar{\mu}_n)^{\alpha-1} - 1} \sum_{i=1}^{n-2} (n - 2) \cdots (n - i - 1) \left( \prod_{k=n-i+1}^{n-1} \frac{\mu_k}{1 + \mu_k} \right) \mu_{n-i} \right].
\] (50)

By the induction hypothesis, we know that for all $1 < i < n + 1$,
\[
\left( \frac{\mu_i}{1 + \mu_i} \right) f_i(\mu) - f_{i+1}(\mu) > 0
\]
when $\mu > \mu_i$. Moreover, since $\frac{(n-1+\alpha)}{n-1} > \frac{1}{n-1+\alpha}$ when $n \geq 3$ and $\left( \frac{1+\bar{\mu}_n}{1+\mu_n} \right)^{\alpha-1} < 1$ for $\bar{\mu}_n \in (\mu_n, \mu)$, we know that (50) is positive.
References


