Are eBay Auctions Efficient?
Sequential Ascending Auctions with New Buyer Entries

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August 18, 2008

Abstract
I model the environment of Internet auction sites, such as eBay, as sequential ascending auctions. New buyers may enter the auction site after some of the auctions have completed and only bid for the remaining auctions. I characterize a perfect Bayesian equilibrium in the dynamic game. Because incumbent buyers have revealed their own valuations in earlier auctions while new entrants do not, information is asymmetric among buyers during the bidding process. Consequently, their bidding strategies are different. A lower-valuation buyer may win an auction while a higher-valuation buyer restrains from bidding higher, resulting an inefficient allocation. In general, the expected transaction prices would increase over time. Comparing to selling the multiple items in a single-round simultaneous auction, sellers can exploit the information asymmetry in sequential auctions to obtain a higher revenue.

1 Introduction

Internet auctions have grown rapidly in the past decade. Although there have been many research papers on Internet auctions\(^1\), one important feature remains lack of attention —

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\(^1\)See Bajari and Hortacsu (2004); Ockenfels, Reiley, and Sadrieh (2006) for surveys on these studies.
entry of new buyers. Contrary to most traditional brick-and-mortar auctions, buyers usually
do not arrive at an Internet auction site simultaneously. For a sequence of auctions, new
buyers may enter the auction site only later in the sequence. This paper proposes a model
to analyze the effect of buyer entries when buyers have affiliated private values.

Most previous research on sequential auctions bases on the framework proposed by Mil-
grom and Weber (2000) and Weber (1983). All buyers arrive before the first auction takes
place. Each buyer has inelastic unit demand. An auction begins after the previous one has
completed. If a buyer does not win in an early auction, she can participate in the next one.
Because of the opportunity to participate in future auctions, buyers would lower their bids in
the early auction. With independent private values, Weber (1983) shows that the expected
transaction price of each auction is the same for both the sequential first-price and second-
price auctions because the effect of winners leaving the auction site and the effect of bidders
shading their bids in early auctions to account for future options exactly cancel out.

In many auction situations, especially in Internet auctions, not all buyers enter an auction
site before the first auction is being conducted. Instead, new buyers are likely to enter the site
and participate only in later auctions. Buyer entry in sequential auctions attracts academic
attention only recently. Huang, Chen, Chen, and Chow (2008) propose a model to analyze
overlapping auctions with deterministic buyer entries. That find that, although the time span
of one auction may overlap with the span of another one, buyers only bid on the auction which
is the first to end among all remaining auctions. Consequently, without loss of generality,
overlapping auctions can be treated as sequentially auctions. Ex ante transaction prices are
identical across auctions in their model. Said (2008) considers stochastic entry in sequential
ascending auctions. He shows that bidding strategies are memoryless. Bids only depend
on the information revealed in the current auction, but not on any information revealed
in earlier auctions. There is no systematic trend in the expected transaction prices. Both
studies assume that consumers have independent private valuations and find the allocation
of sequential ascending auctions to be efficient.

\footnote{This paper was written in 1982, but not published until 2000.}
In this paper, I relax the independence assumption on buyers' valuations. When buyers' valuations are affiliated, the analysis is much more complicated. Milgrom and Weber (2000) find that sequential first-price auctions would have ascending expected prices because buyers are willing to bid more in later auctions after observing the information revealed in earlier auctions. As for sequential second-price auctions, Wang (2006) shows that the expected prices also ascend in a sequential second-price auction with affiliated private values. On the other hand, empirical studies tend to find descending prices across auctions.\footnote{For instance, see Ashenfelter (1989).} In this paper, I show that the transaction prices increases over time if many high-valuation buyers may enter in later auctions. However, if the number of entrants in later auctions is restricted to one, the expected transaction prices would be identical for all auctions.

The model in this paper consists of a sequence of ascending auctions. Each sells an identical good. All buyers have inelastic unit demand for the good. Buyers enter the auction site sequentially and stay until winning a good. I characterize a perfect Bayesian equilibrium in this dynamic game. The allocation of the auction mechanism is not necessarily efficient in equilibrium. It is possible that a buyer with a lower valuation obtains the good while a buyer with a higher valuation does not. Allowing new buyers to enter \textit{per se} does not cause inefficiency in the allocation of the items. However, when buyers’ valuations are affiliated, inefficiency results from the information asymmetry among buyers. My findings are in contract with the efficient equilibrium of asymmetric English auctions found in Maskin (1992) and Krishna (2003, chap. 9). In those papers, they focus on the asymmetric \textit{valuations} among buyers, not the asymmetric \textit{information} among them.

The driving force behind the inefficiency is asymmetric information among buyers.\footnote{Previous research on sequential auctions with asymmetrically informed buyers assumes some buyers are \textit{exogenously} better-informed than other buyers. (Engelbrecht-Wiggans and Weber, 1983; Bennouri and Falconieri, 2006; Hörner and Jamison, 2008) In my model, the information asymmetry is a consequence of buyers’ behavior in earlier stages.} While some buyers have participated in an earlier auction and revealed their own valuations through the bidding behavior, new buyers do not. Since new buyers know their own private valuations, they have better information than old buyers. Because of affiliation in valuations, the asym-
metricaly in turn creates different expectations about the valuations of incoming buyers. New buyers get information rents from their better information and bid 
less aggressively, other things being equal. Hence, the allocation of the sequential auctions is not always efficient.

I also compare selling the items in a single-round multiple-object auction with sequential auctions. When the number of new buyers in later auctions is restrict to one, the expected transaction price under sequential auctions is higher than that under a single-round multiple-object auction. Consequently, from a seller’s point of view, he can exploit the information asymmetry under sequential auctions to increase his revenue. For instance, when there are three items and four buyers, the seller can adopt the following strategy: Hold the first auction after the arrival of the first two buyers, the second auction after the arrival of the third buyer, and the final auction after the arrival of the last buyer.

The rest of the paper is organized as the following. In the next section, I introduce a model of sequential ascending auctions with new buyer entries. A perfect Bayesian equilibrium of the dynamic game is presented in Section 3. In Section 4, I compare sequential auctions with a single-round multiple-object auction. To provide a concrete idea of the model equilibrium, I show an numerical example in Section 5. Concluding remarks are in the final section.

2 Model

Consider an affiliated private value model with three auctions. Under the private value assumption, each buyer has perfect information about her own valuation for the item. Under the affiliated value assumption, buyers’ valuations are positively correlated. When there are only two auctions, the information obtained in the first auction is useless in the second auction because it is a dominant strategy to bid one’s own valuation in the second (final) auction.

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5 Two recent papers compare selling multiple objects in sequential auctions versus a single-round simultaneous auction under the format of sealed-bid uniform-price auction. Mezzetti, Pekeč, and Tsetlin (2008) allow valuations to be affiliated and find the revenue to be higher in a single-round auction if winning bids are not announced, but the comparison is inconclusive if these bids are announced. Feng and Chatterjee (2005) assume that both buyers and the monopolistic seller are impatient. If the number of buyers relative to the number of objects is low enough, the seller may obtain a higher expected payoff when auctions are conducted sequentially. Otherwise, it is better to sell all objects simultaneously.
The simplest model to demonstrate the effect of the asymmetric information caused by buyer entries consists of three auctions.

Following the empirical findings in Zeithammer and Adams (2007), I model eBay auctions as ascending auctions.\(^6\) Everyone knows that three auctions, each with one identical object to be sold, will be held sequentially in periods \(t = 1, 2, 3\), respectively.

Buyers may arrive in any of the three periods. Each buyer has inelastic unit demand. A winning bidder of an auction leaves immediately and does not participate in future auctions. On the contrary, buyers who lose in an early auction can always participate in the next one.

Let \(L_t\) denote the number of buyers who arrive in period \(t\) (new buyers) and \(M_t\) denote the number of buyers who have arrived before period \(t\) (old buyers) for \(t = 1, 2, 3\). By construction, \(M_1 = 0\) in the first period and \(M_t = M_{t-1} + L_{t-1} - 1\) for \(t = 2, 3\). The number of new buyers in each period is exogenous and commonly known.

A buyer’s valuation is a random variable drawn from the support \([0, v]\). Its distribution is continuous on \((0, v]\), but I allow the probability mass to be positive at the lower bound 0. My model can account for stochastic entry of buyers as long as I interpret entry as the random event of drawing a positive valuation and no entry as the random event of drawing a zero valuation. For buyers entering in the same period, the distribution of their valuations is symmetric. Let \(V_{oi}^t\) denote the \(i\)-th highest valuation among all old buyers in period \(t\) and \(V_{ni}^t\) denote the \(i\)-th highest valuation among all new buyers. The realized values are expressed as the lower cases, \(v_{oi}^t\) and \(v_{ni}^t\), respectively. The distribution functions are denoted as \(F_{oi}^t\) and \(F_{ni}^t\). The joint distribution of the valuations is common knowledge among all buyers, but the realized values are private. The valuations are affiliated in the sense that the joint distribution of valuations satisfies Assumption 1. This assumption requires that knowing one buyer’s valuation has a non-negative effect on the expected value of another buyer’s own valuation, but the influence is not too large.

\begin{assumption}
The expected value of any buyer’s valuation weakly increases in another
\end{assumption}

\(^6\)They empirically test the bidding behavior on eBay and find the data are better described by ascending auctions rather than second-price sealed-bid auctions.
buyer’s valuation by a factor less than one.

\[ 0 \leq \frac{d}{dv_j} E [V_i | V_j = v_j, \Omega] < 1 \]

for any \( j \neq i \) and any public information \( \Omega \).

It is costless to bid. There is no discount between periods. Every buyer is risk-neutral.

A buyer’s surplus in the game is her valuation \( v_i \) less the transaction price if she wins in one of the auctions. The surplus is zero otherwise. A buyer wants to maximize her expected surplus.

Each auction is conducted as a “button auction”. All buyers participating in that period press on a button at the beginning of a period. The standing price starts from zero and keeps rising.\(^7\) A buyer depresses the button once she decides to quit from the current auction. When a buyer quits from one auction, she cannot participate in the same auction again. The standing price rises until only one buyer remains. The last remaining buyer wins the auction and pays the final standing price. A buyer’s dropout price and her identify are both observed by all buyer, including buyers who enter in future periods.

Since buyers’ valuations are random variables, the transaction prices are also random variables. Let \( P_t^* \) denote the transaction price of the auction in period \( t \). Its realized value is represented by the lowercase \( p_t^* \).

3 Perfect Bayesian Equilibrium

In this section, I demonstrate a symmetric perfect Bayesian equilibrium of this three-period auction game. The equilibrium is symmetric in the sense that buyers would adopt the same bidding strategy if they have identical private valuation and identical information. In equilibrium, each buyer’s dropout price strictly increases in her valuation for any given public information. Hence, every buyer uses the dropout prices to infer others’ valuations. An old

\(^7\)The seller cannot choose the starting point of the standing price nor a hidden reserve price. See Caillaud and Mezzetti (2004) for discussion on sequential ascending auctions with reserve prices.
buyer’s valuation is revealed before the current period begins.

Although an old buyer’s valuation can also be inferred from her dropout price in the current period, the inferred valuation is identical to the one inferred from the dropout price in the previous period in the equilibrium path. I assume that, in the off-equilibrium event of observing conflicted inferred valuations from different periods for a particular buyer, all other buyers’ belief is determined by the valuation revealed in an earlier period.

Because the current standing price can be used to infer a lower bound for a remaining new buyer’s valuation, the common belief of any buyer at a given time consists of (i) the exact valuations of all old buyers, (ii) the exact valuations of all new buyers who have dropped out, and (iii) the lower bound for the valuations of all remaining new buyers.

3.1 Final Auction

As is well-known in the literature, a buyer’s dominant strategy in an ascending auction is to bid until her own private valuation. The bidding strategy of a buyer with valuation $v$ is $\beta_3(v) = v$. This strategy is independent of information. The information revealed in earlier auctions does not play any role in the final auction.

The bidding strategy is the same regardless whether the buyer is an old buyer or a new one. As a result, the winner of Auction Three is the buyer with the highest valuation in this period. The transaction price is equal to the second highest valuation among the active buyers in Period Three.

$$p_3^* = \begin{cases} v_{32}^n, & \text{if } v_{32}^n \geq v_{31}^o; \\ v_{31}^n, & \text{if } v_{31}^n \geq v_{31}^o \geq v_{32}^o; \\ v_{31}^o, & \text{if } v_{31}^o \geq v_{31}^o \geq v_{32}^n; \\ v_{32}^o, & \text{if } v_{32}^o \geq v_{31}^o. \end{cases}$$
3.2 Second Auction

3.2.1 Bidding Strategies

I will claim that the bidding strategies satisfy the following monotonicity property: “Given the number of new buyers remaining bidding in Auction Two, each new buyer’s maximal amount to bid strictly increases in her own valuation. When the buyer with the lowest valuation among remaining new buyers quits, she does not induce the immediate quit of any other buyer.”

Suppose the monotonicity property holds in the case with \((i + 1)\) new buyers remaining bidding. I will show that the monotonicity property holds for the case with \(i\) new buyers. When the \((L_2 - i)\)-th new buyer drops from Auction Two, there are \(i\) new buyers remaining bidding. Given the monotonicity property in the case with \((i + 1)\) new buyers, the valuations of those who have dropped are publicly known. The public information is

\[
\Omega_{2i} \equiv \{V_{2i}^n \geq v_{2,i+1}^n; V_{2k}^n = v_{2k}^o, k = i + 1, i + 2, \ldots, L_2; V_{2k}^o = v_{2k}^o, k = 1, 2, \ldots, M_2\},
\]

where \(v_{2,i+1}^n\) is inferred from the \((L_2 - i)\)-th new buyer’s dropout price. For a new buyer with valuation \(v\), denote her maximal amount to bid as \(\beta_n^2(v; \Omega_{2i})\). At this moment, I simply assume that \(\beta_n^2(v; \Omega_{2i})\) is an increasing function of \(v\), but do not define its value. In the following paragraphs, I will show that, under the belief that all other new buyers follow the monotonic strategy \(\beta_n^2(\cdot; \Omega_{2i})\), each remaining new buyer would adopt a monotonic bidding strategy. Besides, I will derive the value of the bidding function.

To consider a buyer’s maximal bidding price, we need to compute her expected surplus of quitting the current auction and participating in the next auction. A buyer obtains positive surplus from the next auction only if she has the highest valuation among all the losers of Auction Two and her valuation is more than the highest valuation among new buyer in Auction Three. Denote the \(i\)th highest valuation, including both old and new buyers, in Auction Two as \(V_{(i)}\). Under the belief that other new buyers follow the symmetric bidding strategy \(\beta_n^2\), the next new buyer to quit from Auction Two must be the one with valuation \(v_{2i}^o\),
the lowest valuation among the remaining new buyers. Consequently, she can infer \( V_{2i}^n = v \).

Her expected surplus of being the next buyer to quit among all remaining buyers is

\[
S_2^n(v; \Omega_{2i}) = E \left[ \max \{ v - \max \{ V_{(3)}, V_{31}^n \}, 0 \} \mid V_{2i}^n = v, \Omega_{2i} \right]. 
\]  

(1)

**Lemma 1.** Given the public information \( \Omega_{2i} \), the derivative of the expected surplus of the future auction conditional on being the first buyer to drop is between zero and one for new buyers. For any \( v \in (0, \bar{v}) \)

\[
0 \leq \frac{dS_2^n(v; \Omega_{2i})}{dv} < 1.
\]

**Proof.** By Assumption 1, affiliation implies

\[
0 \leq \frac{dE[\max \{ V_{(3)}, V_{31}^n \} \mid V_{2i}^n = v, \Omega_{2i}]}{dv} < 1.
\]

Therefore, from equation (1), I obtain \( 0 \leq dS_2^n(v; \Omega_{2i})/dv \leq 1 \). In addition, since \( \Pr(v < \max \{ V_{(3)}, V_{31}^n \} \mid V_{2i}^n = v, \Omega_{2i}) > 0 \) for any \( v < \bar{v} \), the derivative of \( S_2^n(v; \Omega_{2i}) \) with respect to \( v \) is less than one. Therefore, \( 0 \leq dS_2^n(v; \Omega_{2i})/dv < 1 \) \( \square \)

The surplus of winning Auction Two at a standing price \( p_2 \) is simply \( v - p_2 \). The expected surplus of quitting Auction Two is \( S_2^n(v; \Omega_{2i}) \). As a result, a new buyer prefers quitting the auction to staying if and only if \( S_2^n(v; \Omega_{2i}) \geq v - p_2 \), which is equivalent to \( p_2 \leq v - S_2^n(v; \Omega_{2i}) \).

The maximal amount to bid in Auction Two is thus

\[
\beta_2^n(v; \Omega_{2i}) = v - S_2^n(v; \Omega_{2i}).
\]  

(2)

**Proposition 1.** A new buyer’s bidding function, \( \beta_2^n(v; \Omega_{2i}) \), is an increasing function of \( v \).

**Proof.** The partial derivative of \( \beta_2^n(v; \Omega_{2i}) \) with respect to \( v \) is \( 1 - dS_2^n(v; \Omega_{2i})/dv \), which lies in the interval \( (0, 1] \) according to Lemma 1. \( \square \)

**Lemma 2.** When the buyer with the lowest valuation among remaining new buyers quits from Auction Two, it does not induce the immediate quit of any other new buyer.
Proof. For any new buyer with valuation \( v > v_{2i}^n \),

\[
\beta^n_2(v; \Omega_{2,i-1}) = E[\min\{\max\{V(3), V_{31}^n\}, v\}|V_{2,i-1}^n = v, \Omega_{2,i-1}] \\
> E[\min\{\max\{V(3), V_{31}^n\}, v_{2i}^n\} | \Omega_{2,i-1}] = \beta^n_2(v_{2i}^n; \Omega_{2i}).
\]

Consequently, no other new buyer would quit immediately after the buyer with valuation \( v_{2i}^n \) quits. \( \square \)

Proposition 1 shows that, when there are \( i \) new buyers remaining bidding in Auction Two, every new buyer’s bidding function is monotonic under the belief that all other new buyers use a monotonic bidding strategy. Therefore, the optimal strategy is consistent with the belief. Combining Proposition 1 and Lemma 2, the monotonicity property stated at the beginning of this subsection holds for the case with \( i \) new buyers remaining. By induction, the monotonicity property holds for any number of new buyers remaining bidding in Auction Two.

Because of the monotonicity of \( \beta^n_2 \), old buyers can infer a lower bound of \( V_{2i}^n \) from the current standing price \( p_2 \) by its inverse function. Denote the inferred lower bound as

\[
\hat{v}(\cdot; \Omega_{2i}) \equiv [\beta^n_2]^{-1}(\cdot; \Omega_{2i}).
\]

For the old buyer with valuation \( v_{2k}^o \), the expected surplus of being the next one to drop from Auction Two is

\[
S^o_2(v_{2k}^o, p_2; \Omega_{2i}) = E\left[\max\{v_{2k}^o - \max\{V(3), V_{31}^n\}, 0\}|V_{2i}^n \geq \hat{v}(p_2; \Omega_{2i}), \Omega_{2i}\right]. \tag{3}
\]

Lemma 3. The next old buyer to quit must be the one with the lowest valuation, \( v_{2j}^o \).

Proof. Consider the valuations of any two old buyers with \( v_{2k}^o > v_{2k'}^o \). Their difference in the surplus of winning the current auction is \( v_{2k}^o - v_{2k'}^o \). On the other hand, because they have the same information set but the probability of winning in the next auction is less than one, the
difference in the expected surplus of quitting the current auction must be less than $v_{2k}^o - v_{2k}$.

Consequently, as long as the buyer with a lower valuation ($v_{2k}^o$) remains bidding in Auction Two, the buyer with a higher valuation ($v_{2k}^o$) must strictly prefer staying to quitting. $\square$

For any given information set $\Omega_{2i}$ and the lowest valuation among remaining older buyers $v_{2j}^o$, define the function $x(v_{2j}^o; \Omega_{2i})$ as the solution of $x$ in the following equation.

$$v_{2j}^o - E\{v_{2j}^o - \max\{V_{(3)}, V_{31}^n\}, 0\}|V_{2i}^n \geq x, \Omega_{2i}\} = x - E\{x - \max\{V_{(3)}, V_{31}^n\}, 0\}|V_{2i}^n = x, \Omega_{2i}\}$$

(4)

I will show that the value of $x(v_{2j}^o; \Omega_{2i})$ determines whether the next buyer to drop from Auction Two is a new buyer or an old one.

To show that $x(v_{2j}^o; \Omega_{2i})$ is a well-defined monotonic function, I impose the following assumption.

**Assumption 2.** The affiliation among buyers’ valuations is not too large so that the following two conditions hold.

$$\frac{\partial}{\partial v} E\{\min\{\max\{V_{(3)}, V_{31}^n\}, v\}|V_{2i}^n = v, \Omega_{2i}\} > \frac{\partial}{\partial v} E\{\min\{\max\{V_{(3)}, V_{31}^n\}, v_{2j}^o\}|V_{2i}^n \geq v, \Omega_{2i}\}$$

(5)

$$\frac{\partial}{\partial v_{2j}^o} E\{\min\{\max\{V_{(3)}, V_{31}^n\}, v_{2j}^o\}|V_{2i}^n \geq x, \Omega_{2i}\} > \frac{\partial}{\partial v_{2j}^o} E\{\min\{\max\{V_{(3)}, V_{31}^n\}, x\}|V_{2i}^n = x, \Omega_{2i}\}$$

(6)

**Lemma 4.** Under Assumption 2, the function $x(v_{2j}^o; \Omega_{2i})$ is well-defined and monotonic.

**Proof.** I will prove the existence of a solution to (4) by the intermediate value theorem. Equation (4) can be alternatively expressed as

$$E\{\min\{\max\{V_{(3)}, V_{31}^n\}, v_{2j}^o\}|V_{2i}^n \geq x, \Omega_{2i}\} - E\{\min\{\max\{V_{(3)}, V_{31}^n\}, x\}|V_{2i}^n = x, \Omega_{2i}\} = 0$$

(7)

When $x = v_{2j}^o$, the first term is greater than or equal to the second term by the affiliation between $V_{2i}^n$ and $\max\{V_{(3)}, V_{31}^n\}$. On the other hand, when $x = \pi$, the first term is less than
or equal to second term because \( v_{2j}^o \leq v \). Consequently, there exists a value \( x \in [v_{2j}^o, v] \) such that equation (7) holds.

By the assumption in (5), the partial derivative of (7) with respect to \( x \) is negative. Therefore, there must be a unique solution in the equation. The function \( x(v_{2j}^o; \Omega_{2i}) \) is well-defined. Furthermore, by the assumption in (6), the partial derivative of (7) with respect to \( v_{2j}^o \) is positive. According to the implicit function theorem, the function \( x(v_{2j}^o; \Omega_{2i}) \) strictly increases in \( v_{2j}^o \).

Lemma 5. The old buyer with valuation \( v_{2j}^o \) keeps bidding in Auction Two until the standing price reaches the maximal amount bid by a new buyer with valuation \( x(v_{2j}^o; \Omega_{2i}) \). Consequently, the next buyer to drop from Auction Two is a new buyer if and only if \( v_{2i}^n < x(v_{2j}^o; \Omega_{2i}) \).

Proof. The surplus of winning Auction Two at price \( p_2 \) is \( v_{2j}^o - p_2 \). It suffices to show that \( v_{2j}^o - p_2 \geq S_2^o(v_{2j}^o, p_2; \Omega_{2i}) \) if and only if \( p_2 \leq \beta_2^n(x(v_{2j}^o; \Omega_{2i}); \Omega_{2i}) \). Substituting the variable \( p_2 \) by \( \hat{v} \) using \( p_2 = \beta_2^n(\hat{v}; \Omega_{2i}) \), it is equivalent to show that

\[
v_{2j}^o - \beta_2^n(\hat{v}; \Omega_{2i}) \geq S_2^o(v_{2j}^o, \beta_2^n(\hat{v}; \Omega_{2i}); \Omega_{2i}) \quad \text{if and only if} \quad \beta_2^n(\hat{v}; \Omega_{2i}) \leq \beta_2^n(x(v_{2j}^o; \Omega_{2i}); \Omega_{2i}).
\] (8)

Since \( \beta_2^n \) is monotonic, the last expression is in turn equivalent to \( \hat{v} \leq x(v_{2j}^o; \Omega_{2i}) \). Since \( S_2^o(v_{2j}^o, \beta_2^n(\hat{v}; \Omega_{2i}); \Omega_{2i}) = E[\max\{v_{2j}^o - \max\{V(3), V_{31}^n\}, 0\}|\Omega_{2i}] \) and \( \beta_2^n(\hat{v}; \Omega_{2i}) = \hat{v} - E[\max\{\hat{v} - \max\{V(3), V_{31}^n\}, 0\}|\Omega_{2i}] \), the statement in (8) is true by Lemma 4.

The bidding strategy of the old buyer with valuation \( v_{2j}^o \) is

\[
\beta_2^n(v_{2j}^o; \Omega_{2i}) = \beta_2^n(x(v_{2j}^o; \Omega_{2i}); \Omega_{2i}).
\] (9)

It is also the solution \( p_2 \) to the following equation.

\[
v_{2j}^o - p_2 = S_2^o(v_{2j}^o, p_2; \Omega_{2i}).
\]
3.2.2 Transaction Results

Auction Two ends when there is only one buyer remaining. The transaction price is determined by the standing price at which the last buyer decides to drop from this auction. To find out the winner and her transaction price, we need to figure out who are the last two buyers to remain bidding in Auction Two. There are four possible cases:

- $\beta^o_2(v_{o2}^j; \Omega_{2i}) > \beta^o_2(v_{o2}^i; \Omega_{2i})$: The new buyer with $v_{o2}^j$ is the winner, and the price is $\beta^o_2(v_{o2}^j; \Omega_{2i})$.

- $\beta^o_2(v_{o2}^i; \Omega_{2i}) > \beta^o_2(v_{o2}^j; \Omega_{2i})$ and $\beta^o_2(v_{o2}^j; \Omega_{2i}) > \beta^o_2(v_{o2}^i; \Omega_{2i})$: The old buyer with $v_{o2}^j$ is the winner, and the price is $\beta^o_2(v_{o2}^j; \Omega_{2i})$.

- $\beta^o_2(v_{o2}^i; \Omega_{2i}) > \beta^o_2(v_{o2}^j; \Omega_{2i})$ and $\beta^o_2(v_{o2}^j; \Omega_{2i}) > \beta^o_2(v_{o2}^i; \Omega_{2i})$: The new buyer with $v_{o2}^j$ is the winner, and the price is $\beta^o_2(v_{o2}^j; \Omega_{2i})$.

- $\beta^o_2(v_{o2}^i; \Omega_{2i}) > \beta^o_2(v_{o2}^j; \Omega_{2i})$: The old buyer with $v_{o2}^i$ is the winner, and the price is $\beta^o_2(v_{o2}^i; \Omega_{2i})$.

**Proposition 2.** Given any number of remaining new buyers $i$, the maximal amount the old buyer with valuation $v_{o2}^i$ is willing to bid in Auction Two is higher than the amount a new buyer with the same valuation is willing to bid.

$$\beta^o_2(v_{o2}^i; \Omega_{2i}) \geq \beta^n_2(v_{o2}^i; \Omega_{2i}).$$

The equality holds if and only if $\max\{V_{3j}, V_{31}^n\}$ conditioning on $\Omega_{2i}$ is mean-independent of $V^o_{2i}$.

**Proof.** Fix $v_{o2}^i$ in the proof. Since $\beta^o_2(v_{o2}^i; \Omega_{2i})$ and $\beta^n_2(v_{o2}^j; \Omega_{2i})$ are solutions of $p_2$ to $v_{o2}^j - p_2 = S^n_2(v_{o2}^j; \Omega_{2i})$ and $v_{o2}^j - p_2 = S^n_2(v_{o2}^j, p_2; \Omega_{2i})$, respectively, it suffices to show that $S^n_2(v_{o2}^j, p_2; \Omega_{2i}) \leq S^n_2(v_{o2}^j; \Omega_{2i})$ at $p_2 = \beta^n_2(v_{o2}^j; \Omega_{2i})$. A new buyer knows her own private valuation $V_{2i}^n = v_{o2}^j$ while an old buyer only knows a lower bound for the random variable
By the affiliation between \( \max\{V(3), V_{31}^n\} \) and \( V_{2i}^n \), we have

\[
S_2^o(v; \beta_2^o(v_{2j}; \Omega_{2i}); \Omega_{2i}) = E \left[ \max\{v_{2j}^o - \max\{V(3), V_{31}^n\}, 0\} \mid V_{2i}^n \geq v_{2j}^o, \Omega_{2i} \right]
\]

\[
\leq E \left[ \max\{v_{2j}^o - \max\{V(3), V_{31}^n\}, 0\} \mid V_{2i}^n = v_{2j}^o, \Omega_{2i} \right]
\]

\[
= S_2^o(v_{2j}^o; \Omega_{2i}).
\]

The equality holds if and only if \( \max\{V(3), V_{31}^n\} \) conditioning on \( \Omega_{2i} \) is mean-independent of \( V_{2i}^n \).

Proposition 2 implies that \( x(v_{21}^o; \Omega_{21}) \geq v_{21}^o \). Consequently, when the last two buyers remaining in Auction Two consists of one new buyer and one old buyer, the allocation of the item may be inefficient.

**Proposition 3.** When the valuations \( V_{31}^n \) and \( V_{21}^n \) are affiliated, a buyer may win the item sold in Auction Two even if her valuation is not the highest one. On the other hand, when \( V_{31}^n \) and \( V_{21}^n \) are independent, the item is always sold to the buyer with the highest valuation.

**Proof.** Consider the case in which the last two buyers in Auction Two consists of one new buyer and one old buyer. Note that when the new buyer with valuation \( v_{22}^o \) drops, the public information set is \( \Omega_{21} = \{V_{21}^n \geq v_{22}^o, V_{2k}^o = v_{2k}^o, k = 2, 3, \ldots, L_2; V_{2k}^o = v_{2k}^o, k = 1, 2, \ldots, M_2\} \).

Therefore, the realized value of \( V(3) \) is known given the information set \( \Omega_{21} \). By Proposition 2, \( \beta_2^o(v_{21}^o; \Omega_{21}) \geq \beta_2^o(v_{21}^o; \Omega_{21}) \). When the inequality holds strictly, a new buyer with a higher valuation than an old buyer \( (v_{21}^o > v_{21}^o) \) may drop from Auction Two earlier \( (\beta_2^o(v_{21}^o; \Omega_{21}) < \beta_2^o(v_{21}^o; \Omega_{21})) \).

The inequality in Proposition 2 is strict if the conditional mean of \( \max\{V(3), V_{31}^n\} \) given \( \Omega_{21} \) depends on \( V_{21}^n \). Since \( V(3) \) is non-random conditioning on \( \Omega_{21} \), this condition is equivalent to “The mean of \( V_{31}^n \) conditioning on \( \Omega_{21} \) depends on \( V_{21}^n \).” Consequently, when \( V_{31}^n \) and \( V_{21}^n \) are affiliated, an old buyer may win Auction Two even if her valuation is not the highest among all buyers active in Auction Two. On the other hand, when \( V_{31}^n \) and \( V_{21}^n \) are independent, the conditional mean of \( \max\{V(3), V_{31}^n\} \) given \( \Omega_{21} \) does not depends on \( V_{21}^n \), and the winner
of Auction Two must be the buyer with the highest valuation.

Before proceeding to the analysis of the first auction, define the following notations for the four possible outcomes of Auction Two.

\[
A_1 \equiv \{\beta^o_2(V^o_n; \Omega^o_{22}) > \beta^o_2(V^o_o; \Omega^o_{21})\}
\]

\[
A_2 \equiv \{\beta^o_2(V^o_n; \Omega^o_{21}) > \beta^o_2(V^o_o; \Omega^o_{21}), \beta^o_2(V^o_n; \Omega^o_{21}) > \beta^o_2(V^o_o; \Omega^o_{21})\}
\]

\[
A_3 \equiv \{\beta^o_2(V^o_n; \Omega^o_{21}) > \beta^o_2(V^o_o; \Omega^o_{21}), \beta^o_2(V^o_o; \Omega^o_{22}) > \beta^o_2(V^o_o; \Omega^o_{22})\}
\]

\[
A_4 \equiv \{\beta^o_2(V^o_o; \Omega^o_{21}) > \beta^o_2(V^o_n; \Omega^o_{21})\}
\]

**Lemma 6.** (a) When the new buyer with valuation \(v^o_{22}\) drops earlier than the old buyers with valuations \(v^o_{21}\) in Auction Two (case \(A_3\)), \(v^o_{22} \leq v^o_{21}\) must hold. (b) When the new buyer with valuation \(v^o_{21}\) drops earlier than the old buyers with valuations \(v^o_{22}\) in Auction Two (case \(A_4\)), \(v^o_{21} \leq v^o_{22}\) must hold.

**Proof.** I will prove by contradiction to show that \(v^o_{22} > v^o_{21}\) never occurs in the case \(A_3\). Suppose that the contrary is true. Then it is possible to have \(v^o_{22} > v^o_{21}\) and \(\beta^o_2(v^o_{22}; \Omega_{22}) < \beta^o_2(v^o_{21}; \Omega_{22})\). Because \(V^o_{(3)} \geq V^o_{21}\), I known \(\beta^o_2(v^o_{22}; \Omega_{22}) \geq \beta^o_2(v^o_{21}; \Omega_{22})\). Hence \(\beta^o_2(v^o_{22}; \Omega_{22}) \geq \beta^o_2(v^o_{21}; \Omega_{22})\). This is a contradiction. The proof for part (b) is exactly the same after replacing \(v^o_{22}\), \(v^o_{21}\), and \(\Omega_{22}\) by \(v^o_{21}\), \(v^o_{22}\), and \(\Omega_{21}\), respectively.

### 3.3 First Auction

The analysis of Auction One is similar to that of Auction Two. Nonetheless, I need to account for the effect of information revelation because it would affect other buyers’ bidding strategies in the second auction. I will show that, conditional on the number of remaining bidders, the bidding strategy is monotonic, and the exit of one buyer does not induce an immediate exit of any other buyer. Consequently, except the winner of Auction One, all buyers entering in Auction One reveal their valuations after the auction ends.
All buyers in the first auction are new buyers. Their valuations are private information before the auction starts. However, since each buyer’s expected surplus of participating in future auctions depends on incoming buyers’ valuations, Auction One can be viewed as an auction with a common component in buyers’ valuations. The equilibrium strategy discussed in this subsection is essentially identical to the one discussed in Krishna (2002, pp 90 – 92).

Suppose that $\beta^n_1(v; \Omega_{1i})$ is a monotonic strategy used by a buyer with valuation $v$ when $i$ buyers remain active, where $\Omega_{1i} \equiv \{ V_{1i} \geq v_{1,i+1} ; V_{1j} = v_{1j}, j = i + 1, i + 2, \ldots, L_1 \}$ is the public information when the buyer with valuation $v_{n_1,i+1}$ drops. I will show that, given the belief that other buyers follow a monotonic strategy $\beta^n_1(\cdot; \Omega_{1i})$, a buyer’s maximal amount to bid indeed strictly increases in her own private valuation.

To compute the expected surplus of dropping from Auction One and participating in future auctions, I need to consider the equilibrium in later auctions. First, if the buyer wins Auction Two (i.e. random events $A_2$ or $A_4$), the transaction price would equal to $\beta^n_2(\max\{V^n_{12}, V^n_{13}\}; \Omega_{20})$, which would imply her surplus to be $v - \beta^n_2(\max\{V^n_{12}, V^n_{13}\}; \Omega_{20})$. Second, if she loses in Auction Two (i.e. random events $A_1$ or $A_3$), the expected surplus of participating in Auction Three would be $\max\{v - \max\{V_{(3)}, V^n_{31}\}, 0\}$. The expected surplus of being the next one to drop from Auction One is

$$S^n_1(v; \Omega_{1i}) = E \left[ \max \left\{ \max \{v - \max\{V_{(3)}, V^n_{31}\}, 0\}, v - \beta^n_2(\max\{V^n_{12}, V^n_{13}\}; \Omega_{20}) \right\} \mid V^n_{1i} = v, \Omega_{1i} \right]$$  \hspace{1cm} (10)

**Lemma 7.** For any given public information $\Omega_{1i}$, the derivative of expected future surplus is between zero and one for any buyer. For any $v \in (0, \overline{v})$,

$$0 \leq \frac{dS^n_1(v; \Omega_{1i})}{dv} < 1.$$  

---

In equilibrium, the buyer with $v^n_{1i}$ wins Auction One, and $v^n_{1j}$ becomes $v^n_{2,j-1}$ for $j = 2, 3, \ldots, L_1$. For the last remaining new buyer in Auction Two, because she knows her own valuation $v^n_{21}$, knowing the public information $\Omega_{21}$ allows her to know the valuations of all buyers in Auction Two $\Omega_{20}$. This implies $\beta^n_2(V^n_{21}; \Omega_{21}) = \beta^n_2(V^n_{21}; \Omega_{20})$. By Proposition 2 and Lemma 6, the transaction price is $\beta^n_2(V^n_{21}; \Omega_{20})$ if $V^n_{21} > V^n_{13}$ and $\beta^n_2(V^n_{31}; \Omega_{20})$ otherwise. Besides, given $\Omega_{20}$, all buyers in Auction Two have the same information. Therefore, the bidding function for an old buyer is the same as that for a new buyer under the information $\Omega_{20}$. I can express the transaction price as $\max\{\beta^n_2(V^n_{21}; \Omega_{20}), \beta^n_2(V^n_{31}; \Omega_{20})\} = \beta^n_2(\max\{V^n_{21}, V^n_{13}\}; \Omega_{20})$.  

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Proof. The expected surplus in (10) can be expressed as

\[
S_n(v; \Omega_{1i}) = E \left[ \max \left\{ v - \min \left\{ \max \{V(3), V_{31}^n\}, \beta_2^n \left( \max \{V_{21}^n, V_{13}^n\}; 0 \right) \right\}, 0 \right\} \left| V_{1i}^n = v, \Omega_{1i} \right. \right] \\
= E \left[ \max \left\{ v - \min \left\{ \max \{V(3), V_{31}^n\}, E \left[ \min \left\{ \max \{V(3), V_{31}^n\}, \max \{V_{21}^n, V_{13}^n\} \right\} \Omega_{20} \right] \right\}, 0 \right\} \left| V_{1i}^n = v, \Omega_{1i} \right. \right].
\]

By Assumption 1, the derivative of the above expression with respect to \(v\) is between zero and one. Moreover, the random event “\(v < \min \{\max \{V(3), V_{31}^n\}, \max \{\beta_2^n(V_{21}^n; \Omega_{20}), \beta_2^n(V_{13}^n; \Omega_{20})\}\}” occurs with a positive probability for any \(v < \bar{v}\). Therefore, \(0 \leq dS_n(v; \Omega_{1i})/dv < 1\).

Because a buyer’s dropout price reveals her valuation, which affects other buyers’ bidding behavior in Auction Two, I need to take into account the potential strategic choice of the dropout price. Since \(\beta_1^n\) is monotonic, choosing a dropout price is equivalent to choosing the valuation to be inferred. For a buyer with valuation \(v\), when she chooses to be inferred as valuation \(\tilde{v}\), the expected surplus of being the next buyer to drop is

\[
\tilde{S}_1^n(v, \tilde{v}; \Omega_{1i}) = E \left[ \max \left\{ v - \min \left\{ \max \{V(3), V_{31}^n\}, \beta_2^n \left( \max \{V_{21}^n, V_{13}^n\}; \tilde{\Omega}_{20}(\tilde{v}) \right) \right\}, 0 \right\} \left| V_{1i}^n = v, \Omega_{1i} \right. \right] (11)
\]

where the revealed information set \(\tilde{\Omega}_{20}(\tilde{v})\) is identical to \(\Omega_{20}\) expect replacing \(V_{2,i-1}^o = v\) by \(V_{2,i-1}^o = \tilde{v}\).

**Proposition 4.** Using the bidding function

\[
\beta_1^n(v; \Omega_{1i}) = v - S_1^n(v; \Omega_{1i}) (12)
\]

constitutes a symmetric perfect Bayesian equilibrium in the first period.

Proof. Claim that, under the belief that other buyers use \(\beta_1^n(v; \Omega_{1i}) = v - S_1^n(v; \Omega_{1i})\) to infer valuations, \(v - \beta_1^n(\tilde{v}; \Omega_{1i}) \geq \tilde{S}_1^n(v, \tilde{v}; \Omega_{1i})\) if and only if \(\tilde{v} \leq v\).

Similar to the arguments in the proof of Lemma 7, the expected surplus in (11) can be
written as

\[ \tilde{S}_1^n(v, \tilde{v}; \Omega_{1i}) = E \left[ \max \left\{ v - \min \left\{ \max \{V_3^{(3)}, V_3^{(1)}\}, E \left[ \min \left\{ \max \{V_3^{(3)}, V_3^{(1)}\}, \max \{V_{21}^{(n)}, V_{13}^{(n)}\}\right| \tilde{\Omega}_{20}(\tilde{v}) \right] \right\} \right\} \bigg| V_{1i}^{n} = v, \Omega_{1i} \right]. \]

Assumption 1 implies

\[ 0 \leq \frac{\partial \tilde{S}_1^n(v, \tilde{v}; \Omega_{1i})}{\partial v} < 1 \]

for any given public information \( \Omega_{1i} \) and \( v \in (0, \pi) \). Therefore,

\[ [v - \beta_1^n(\tilde{v}; \Omega_{1i})] - \tilde{S}_1^n(v, \tilde{v}; \Omega_{1i}) = v - \tilde{v} + \tilde{S}_1^n(\tilde{v}, \tilde{v}; \Omega_{1i}) - \tilde{S}_1^n(v, \tilde{v}; \Omega_{1i}) = \int_{\tilde{v}}^{v} \left[ 1 - \frac{\partial \tilde{S}_1^n(x, \tilde{v}; \Omega_{1i})}{\partial x} \right] dx \]

is positive if and only if \( \tilde{v} > v \).

Consequently, if other buyers use \( \beta_1^n(\cdot; \Omega_{1i}) \) defined in (12) to infer a buyer’s valuation from her dropout price, a buyer’s optimal strategy is to keep bidding in Auction One as long as her inferred valuation is less than \( v \). In other words, the maximal amount to bid is \( \beta_1^n(v; \Omega_{1i}) \). Hence, the belief of following the bidding function \( \beta_1^n \) is consistent with a buyer’s equilibrium behavior.

The following proposition verifies the monotonicity of the bidding function \( \beta_1^n(\cdot; \Omega_{1i}) \).

**Proposition 5.** Conditional on the number of buyers remaining bidding, the bidding function of Auction One strictly increases in valuation.

**Proof.** The slope of \( \beta_1^n(\cdot; \Omega_{1i}) \) is \( 1 - dS_1^n(v; \Omega_{1i})/dv \). By Lemma 7, \( \beta_1^n \) is monotonic. 

Similar to the proof of Lemma 2, a buyer’s exit in Auction One does not induce an immediate exit of another buyer. Consequently, each buyer’s valuation can be inferred from observing her dropout price. By induction, the monotonicity property holds for any number of buyers remaining in Auction One.
Lastly, the transaction price of Auction 1 is

\[ p_1^* = \beta_n^1(v_{12}^n; \Omega_{12}). \]

### 3.4 Expected Transaction Prices

The *ex ante* expected transaction prices weakly increase over these three auctions. This result is the same as the finding of ascending transaction prices of sequential sealed-bid auctions in Milgrom and Weber (2000), but for a different reason. In Milgrom and Weber (2000), the price increases due to the information revelation over time. In my model, buyers entering in later auctions do not have the opportunity to participate in earlier auctions. As a result, earlier buyers may take advantage of this opportunity and win the item with a lower price.

**Proposition 6.** The *ex ante* expected transaction prices increases over time.

\[ E[P_1^*] \leq E[P_2^*] \leq E[P_3^*] \]

**Proof.** First, consider the random event \( A_1 \) in Auction Two.

\[
E[P_2^*|A_1] = E[\beta_n^2(v_{22}^n; \Omega_{22})|A_1] = E[\min \{ \max \{ V_{(3)}^n, V_{31}^n \}, V_{22}^n \}|A_1]
\]

\[
= E[V_{(3)}^n | V_{22}^n \geq V_{(3)}^n \geq V_{31}^n, A_1] \Pr(V_{22}^n \geq V_{(3)}^n \geq V_{31}^n | A_1)
\]

\[
+ E[V_{31}^n | V_{22}^n \geq V_{31}^n \geq V_{(3)}^n, A_1] \Pr(V_{22}^n \geq V_{31}^n \geq V_{(3)}^n | A_1)
\]

\[
+ E[V_{22}^n | \max \{ V_{(3)}^n, V_{31}^n \} \geq V_{22}^n, A_1] \Pr(\max \{ V_{(3)}^n, V_{31}^n \} \geq V_{22}^n | A_1)
\]

\[
\leq E[V_{32}^n | V_{32}^n \geq V_{31}^n, A_1] \Pr(V_{32}^n \geq V_{31}^n | A_1) + E[V_{31}^n | V_{31}^n \geq V_{32}^n, A_1] \Pr(V_{31}^n \geq V_{32}^n | A_1)
\]

\[
+ E[V_{31}^n | V_{31}^n \geq V_{32}^n, A_1] \Pr(V_{31}^n \geq V_{32}^n | A_1) + E[V_{32}^n | V_{32}^n \geq V_{31}^n, A_1] \Pr(V_{32}^n \geq V_{31}^n | A_1)
\]

\[
= E[P_3^*|A_1],
\]

where the inequality uses the fact that \( V_{(3)} = V_{32}^o \) and \( V_{22}^n = V_{31}^o \). Similarly, for the case \( A_2 \), we know \( V_{(3)} = V_{32}^o \) and \( V_{21}^o = V_{31}^o \), and hence \( E[P_2^*|A_2] \leq E[P_3^*|A_2] \). For the case \( A_3 \), according to Lemma 6, \( V_{(3)} = V_{32}^o \) and \( V_{21}^o = V_{31}^o \). Therefore, \( E[P_2^*|A_3] \leq E[P_3^*|A_3] \). Applying the same

Next, consider the expected transaction price of Auction One. In equilibrium, $V^*_2 = V^*_1$ and $V^*_2 = V^*_3$. Since $\beta^*_1(V^*_1; \Omega_{12}) = V^*_1 - S^*_1(V^*_1; \Omega_{12})$,

$$E[P^*_1] = E[\beta^*_1(V^*_1; \Omega_{12})] = E\left[\min \left\{ \min \left\{ \max\{V(3), V_{31}\}, V^*_1 \right\}, \beta^*_2(\max\{V^*_1, V^*_3\}; \Omega_{20}) \right\} \right]$$
$$\leq \Pr(A_1)E\left[\beta^*_2(V^*_2; \Omega_{20})|A_1 \right] + \Pr(A_3)E\left[\beta^*_2(V^*_2; \Omega_{20})|A_3 \right]$$
$$+ \Pr(A_2)E\left[\beta^*_2(V^*_2; \Omega_{20})|A_2 \right] + \Pr(A_4)E\left[\beta^*_2(V^*_2; \Omega_{20})|A_4 \right]$$
$$= E[P^*_2].$$

(14)

The intuition for ascending transaction prices is easy. Consider the following example. Suppose buyers with the top two highest valuations enter only in the final auction. Only one of them can win the item and the price of the final auction equals to the second-highest valuation. Nonetheless, earlier auctions have lower transaction prices because their transaction prices are determined by buyers with lower valuations. Late arrival of many high-valuation buyers causes $E[V^*_3|V^*_3 \geq V^*_3, A_1] \geq E[V^*_2|V^*_3 \geq V^*_3, A_1]$ in (13) and $E[\beta^*_2(V^*_2; \Omega_{20})|A_1] \geq E[\beta^*_2(V^*_2; \Omega_{20})|A_1]$ in (14). Therefore, the ex ante expected transaction prices increase over time.

On the contrary, in the absence of late arrival of many high-valuation buyers, the expected transaction prices are identical.

**Proposition 7.** If there is only one new buyer in Auctions Two and one in Auction Three, the ex ante expected transaction prices are identical across auctions.


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4 Single-Round Auction

In this section, I consider an alternative way to sell the items. Instead of auctioning them sequentially, the auction takes place until all buyers have arrived. All items are sold by a single-round simultaneous multiple-object auction. Specifically, there is a single standing price which starts at zero and keeps rising. At any standing price, a buyer may either stay in the auction or drop out. Once dropping out, it is not allowed to return to the auction. The standing price rises until only three buyers remain bidding. Each of these three buyers can get one item and pay the final standing price.

Under the assumption of affiliated private valuation, a buyer knows her personal valuation for sure. A buyer’s dominant strategy is to stay in the auction if and only if the current standing price is less than her valuation. Consequently, the equilibrium transaction price, denoted by $p^{sim}$, equals to the fourth highest valuation among all buyers.

When many new buyers may arrive later in a sequence of auctions, it is not surprising to find that expected aggregated revenue may be higher under a single multiple-object auction than under sequential auctions. This is because for sequential auctions, the item sold in an earlier auction can not be allocated to high-valuation buyers who have not arrived. The following proposition compares the transaction prices of the two auction methods in the absence of the effect due to late arrival of many high-valuation buyers.

**Proposition 8.** Suppose that there is only one new buyer in Auction Two and one in Auction Three. The expected transaction price under sequential auctions is greater than or equal to the expected transaction price under simultaneous auctions. The inequality is strict if and only if the valuation of the new buyer in Auction Two is strictly affiliated with the valuation of the new buyer in Auction Three.
Proof. By Proposition 7, \( E[P^*_1] = E[P^*_2] = E[P^*_3] \). Let \( V[4] \) denote the fourth highest valuation among these \((L_1 + 2)\) buyers \( \{V^n_{1k}, k = 1, 2, \ldots, L_1; V^n_{21}; V^n_{31}\} \). There are three possible scenarios in Auction Two: \( A_2, A_3, \) and \( A_4 \). Recall that \( E[P^*_2|A_2] = E[\beta^n(V^n_{21}; \Omega_{21})|A_2] = E[\min\{\max\{V(3), V^n_{31}\}, V^n_{21}\}|A_2] \). I need to consider two situations under the case \( A_2 \), depending on whether \( V^n_{21} \geq V^n_{o21} \). Because \( E[\min\{\max\{V(3), V^n_{31}\}, V^n_{21}\}|A_2, V^n_{21} \geq V^n_{o21}] \geq E[V[4]|A_2, V^n_{21} \geq V^n_{o21}] \) and \( E[\min\{\max\{V(3), V^n_{31}\}, V^n_{21}\}|A_2, V^n_{21} \leq V^n_{21}] = E[V[4]|A_2, V^n_{21} \leq V^n_{o21}] \), I have \( E[P^*_2|A_2] \geq E[V[4]|A_2] \). In addition, for the other two cases, \( E[P^*_2|A_3] = E[\min\{\max\{V(3), V^n_{31}\}, V^n_{21}\}|A_3] = E[V[4]|A_3] \) and \( E[P^*_2|A_4] = E[\min\{\max\{V(3), V^n_{31}\}, V^n_{o22}\}|A_4] = E[V[4]|A_4] \). Consequently, \( E[P^*_1] = E[P^*_2] = E[P^*_3] \geq E[V[4]|A_2] = E[P^{sim}] \)

This proposition implies that a revenue-maximizing seller should not bundle all items together into a single multiple-object auction. Instead, if items are sold sequentially, the aggregate expected profit would be higher. Even though the allocation of sequential auctions is socially inefficiently, a seller can exploit information asymmetry to gain his revenue.

5 Numerical Example

In this section, I present a simple example to demonstrate the effect of information asymmetry in sequential ascending auctions. Suppose there are two new buyers in Auction One, one new buyer in Auction Two, and one new buyer in Auction Three. In addition, assume that only the valuations of buyers entering in Auctions Two and Three are affiliated, but the valuations of buyers entering in Auction One are independent of any other valuation. Assume that the distribution of each buyer’s valuation is a normal distribution with mean 10 and unit variance,\(^9\), and the correlation between \( V^n_{21} \) and \( V^n_{31} \) is \( \rho \). It is straightforward to verify that Assumptions 1 and 2 are both satisfied.

To find out the critical value to determine the winner of Auction Two \( x(v^n_{21}, \Omega_{21}) \), note

\(^9\)Since valuations are as non-negative, the negative part of the normal distribution, which occurs with a probability less than \( 10^{-23} \), is truncated.
that, by using integration by parts, equations (1) and (3) can be expressed as

\[
S_n(v_n^2; \Omega_{21}) = E[\max\{v_{n21} - V_{n31}, 0\}|V_{n21} = v_{n21}, \Omega_{21}]
= \int_0^{v_{n21}} (v_{n21} - v_{n31}) dF_{n31}(v_{n31}|V_{n21} = v_{n21}) = \int_0^{v_{n21}} F_{n31}(v_{n31}|V_{n21} = v_{n21}) dv_{n31}
\]

and

\[
S_o(v_o^2, p_2; \Omega_{21}) = E[\max\{v_o^2 - V_{n31}, 0\}|V_{n21} = \hat{v}(p_2; \Omega_{21}), \Omega_{21}]
= \int_0^{v_o^2} F_{n31}(v_{n31}|V_{n21} = \hat{v}(p_2; \Omega_{21})) dv_{n31}.
\]

For any given value of \(v_{o21}\), the function \(x(v_{o21}; \Omega_{21})\) is implicitly as the solution \(x\) in the following equations.

\[
v_{o21} - \int_0^{x} F_{n31}(v_{n31}|V_{n21} \geq x) dv_{n31} = x - \int_0^{x} F_{n31}(v_{n31}|V_{n21} = x) dv_{n31}.
\]

Under the independence assumption on \(V_{n11}\) and \(V_{n12}\), the distribution of \(V_{n31}\) does not depend on \(\Omega_{21}\). Consequently, the function \(x(v_{o21}; \Omega_{21})\) is independent of \(\Omega_{21}\).

The curves in Figure 1 shows the graph of \(x(v_{o21}; \Omega_{21})\) for various values of \(\rho\). The item is sold to the new buyer in Auction Two if and only if the realized valuations \((v_o^2, v_{n21})\) is above the curve. When \(\rho = 0\), the graph coincides with the 45\(^\circ\) line. The allocation is always efficient. For positive correlation between \(V_{n21}\) and \(V_{n31}\) \((\rho > 0)\), the graph is above the 45\(^\circ\) line for all \(v_o^2\). As a result, there is a positive probability of inefficient allocations.

The expected transaction price of sequential auctions is slightly higher than that of a single-round simultaneous auction. Figure 2 shows the transaction prices for these two types of auctions at different correlation levels between the two valuations \(V_{n21}\) and \(V_{n31}\). For instance, when \(\rho = 0.67\), the probability of inefficient allocation in Auction Two is about 5.6\%, and the expected revenue under sequential auctions is higher than the revenue under a simultaneous auction by 0.0031 unit, which is 0.31\% of the standard deviation of the valuations. When \(\rho\) is
Figure 1: The minimal valuation for Buyer 3 to win in Auction 2

Figure 2: The expected transaction prices
close to zero, there is no information asymmetry among buyers and the expected transaction prices are the same for these two auction formats. On the other hand, when \( \rho \) is near one, although the new buyer has better information than the old buyer, the expected surplus of participating in the final auction is approaching zero for both buyers. Consequently, the expected transaction prices are also the same for these two auction formats. The gain from auctioning items sequentially is the largest when the correlation is moderate.

6 Conclusion

One important feature of Internet auctions which is ignored in the traditional auction literature is entry of new buyers. In this paper, I show that, when multiple items of an identical good are sold in sequential auctions and buyers have affiliated private valuations on the good, the transaction prices would increase over time, which is due to the effect of late arrival of many new buyers. Moreover, entry would result in inefficient allocation of the goods because of information asymmetry among buyers. Comparing to selling all items in a single-round multiple-object auction, sellers can exploit the information asymmetry to raise expected revenue.

References


